Abstract

We consider unsupervised estimation of mixtures of discrete graphical models, where the class variable corresponding to the mixture components is hidden and each mixture component over the observed variables can have a potentially different Markov graph structure and parameters. We propose a novel approach for estimating the mixture components, and our output is a tree-mixture model which serves as a good approximation to the underlying graphical model mixture. Our method is efficient when the union graph, which is the union of the Markov graphs of the mixture components, has sparse vertex separators between any pair of observed variables. This includes tree mixtures and mixtures of bounded degree graphs. For such models, we prove that our method correctly recovers the union graph structure and the tree structures corresponding to maximum-likelihood tree approximations of the mixture components. The sample and computational complexities of our method scale as poly(p, r), for an r-component mixture of p-variate graphical models. We further extend our results to the case when the union graph has sparse local separators between any pair of observed variables, such as mixtures of locally tree-like graphs, and the mixture components are in the regime of correlation decay.

Keywords: Graphical models, mixture models, spectral methods, tree approximation.

1 Introduction

Graphical models offer a graph-based framework for representing multivariate distributions, where the structural and qualitative relationships between the variables are represented via a graph structure, while the parametric and quantitative relationships are represented via values assigned to different groups of nodes on the graph (Lauritzen, 1996). Such a decoupling is natural in a variety of contexts, including computer vision, financial modeling, and phylogenetics. Moreover, graphical models are amenable to efficient inference via distributed algorithms such as belief propagation (Wainwright and Jordan, 2008). Recent innovations have made it feasible to train these models with low computational and sample requirements in high dimensions (see Section 1.2 for a brief overview).

Simultaneously, much progress has been made in analyzing mixture models (Lindsay, 1995). A mixture model can be thought of as selecting the distribution of the manifest variables from a fixed
set, depending on the realization of a so-called choice variable, which is latent or hidden. Mixture models have widespread applicability since they can account for changes in observed data based on hidden influences. Recent works have provided provable guarantees for learning high-dimensional mixtures under a variety of settings (See Section 1.2).

In this paper, we consider mixtures of (undirected) graphical models, which combines the modeling power of the above two formulations. These models can incorporate context-specific dependencies, where the structural (and parametric) relationships among the observed variables can change depending on a hidden context. These models allow for parsimonious representation of high-dimensional data, while retaining the computational advantage of performing inference via belief propagation and its variants. The current practice for learning mixtures of graphical models (and other mixture models) is based local-search heuristics such as expectation maximization (EM). However, EM scales poorly in the number of dimensions, suffers from convergence issues, and lacks theoretical guarantees.

In this paper, we propose a novel approach for learning graphical model mixtures, which offers a powerful alternative to EM. At the same time, we establish theoretical guarantees for our method for a wide class of models, which includes tree mixtures and mixtures over bounded degree graphs. Previous theoretical guarantees are mostly limited to mixtures of product distributions (see Section 1.2). These models are restrictive since they posit that the manifest variables are related to one another only via the latent choice variable, and have no direct dependence otherwise. Our work is a significant generalization of these models, and incorporates models such as tree mixtures and mixtures over bounded degree graphs.

Our approach aims to approximate the underlying graphical model mixture with a tree-mixture model. In our view, a tree-mixture approximation offers good tradeoff between data fitting and inferential complexity of the model. Tree mixtures are attractive since inference reduces to belief propagation on the component trees (Meila and Jordan, 2001). Tree mixtures thus present a middle ground between tree graphical models, which are too simplistic, and general graphical model mixtures, where inference is not tractable, and our goal is to efficiently fit the observed data to a tree mixture model.

1.1 Summary of Results

We propose a novel method for learning discrete graphical mixture models. It combines the techniques used in graphical model selection based on conditional independence tests, and the spectral decomposition methods employed for estimating the parameters of mixtures of product distributions. Our method proceeds in three main stages: graph structure estimation, parameter estimation, and tree approximation.

In the first stage, our algorithm estimates the union graph structure, corresponding to the union of the Markov graphs of the mixture components. We propose a rank criterion for classifying a node pair as neighbors or non-neighbors in the union graph of the mixture model, and can be viewed as a generalization of conditional-independence tests for graphical model selection (Anandkumar et al., 2012c; Bresler et al., 2008; Spirtes and Meek, 1993). Our method is efficient when the union graph has sparse separators between any node pair, which holds for tree mixtures and mixtures of bounded degree graphs. The sample complexity of our algorithm is logarithmic in the number of nodes. Thus, our method learns the union graph structure of a graphical model mixture with similar guarantees as graphical model selection (i.e., when there is a single graphical model).

In the second stage, we use the union graph estimate \( \hat{G}_U \) to learn the pairwise marginals of
the mixture components. Since the choice variable is hidden, this involves decomposition of the observed statistics into component models. We leverage on the spectral decomposition method developed for learning mixtures of product distributions (Anandkumar et al., 2012a; Chang, 1996; Mossel and Roch, 2006). In a mixture of product distributions, the observed variables are conditionally independent given the hidden class variable. We adapt this method to our setting as follows: we consider different triplets over the observed nodes and condition on suitable separator sets (in the union graph estimate \( \hat{G}_d \)) to obtain a series of mixtures of product distributions. Thus, we obtain estimates for pairwise marginals of each mixture component (and in principle, higher order moments) under some natural non-degeneracy conditions. In the final stage, we find the best tree approximation to the estimated component marginals via the standard Chow-Liu algorithm (Chow and Liu, 1968). The Chow-Liu algorithm produces a max-weight spanning tree using the estimated pairwise mutual information as edge weights. We establish that our algorithm recovers the correct tree structure corresponding to maximum-likelihood tree approximation of each mixture component. In the special case, when the underlying distribution is a tree mixture, this implies that we can recover tree structures corresponding to all the mixture components. The computational and sample complexities of our method scale as \( \text{poly}(p, r) \), where \( p \) is the number of nodes and \( r \) is the number of mixture components.

Recall that the success of our method relies on the presence of sparse vertex separators between node pairs in the union graph, i.e., the union of Markov graphs of the mixture components. This includes tree mixtures and mixtures of bounded degree graphs. We extend our methods and analysis to a larger family of models, where the union graph has sparse local separators (Anandkumar et al., 2012c), which is a weaker criterion. This family includes locally tree-like graphs (including sparse random graphs), and augmented graphs (e.g., small-world graphs where there is a local and a global graph). The criterion of sparse local separation significantly widens the scope, and we prove that our methods succeed in these models, when the mixture components are in the regime of correlation decay (Anandkumar et al., 2012c). The sample and computational complexities are significantly improved for this class, since it only depends on the size of local separators (while previously it depended on the size of exact separators).

Our proof techniques involve establishing the correctness of our algorithm (under exact statistics). The sample analysis involves careful use of spectral perturbation bounds to guarantee success in finding the mixture components. In addition, for the setting with sparse local separators, we incorporate the correlation decay rate functions of the component models to quantify the additional distortion introduced due to the use of local separators as opposed to exact separators. One caveat of our method is that we require the dimension of the node variables \( d \) to be larger than the number of mixture components \( r \). In principle, this limitation can be overcome if we consider larger (fixed) groups of nodes and implement our method. Another limitation is that we require full rank views of the latent factor for our method to succeed. However, this is also a requirement imposed for learning mixtures of product distributions. Moreover, it is known that learning singular models, i.e., those which do not satisfy the above rank condition, is at least as hard as learning parity with noise, which is conjectured to be computationally hard (Mossel and Roch, 2006). Another restriction is that we require the presence of an observed variable, which is conditionally independent of all the other variables, given the latent choice variable. However, note that this is significantly weaker than the case of product mixture models, where all the observed variables are required to be conditionally independent given the latent factor. To the best of our knowledge, our work is the first to provide provable guarantees for learning non-trivial graphical mixture models (which
are not mixtures of product distributions), and we believe that it significantly advances the scope, both on theoretical and practical fronts.

### 1.2 Related Work

Our work lies at the intersection of learning mixture models and graphical model selection. We outline related works in both these areas.

**Overview of Mixture Models:** Mixture models have been extensively studied (Lindsay, 1995) and are employed in a variety of applications. More recently, the focus has been on learning mixture models in high dimensions. There are a number of recent works dealing with estimation of high-dimensional Gaussian mixtures, starting from the work of Dasgupta (1999) for learning well-separated components, and most recently by (Belkin and Sinha, 2010; Moitra and Valiant, 2010), in a long line of works. These works provide guarantees on recovery under various separation constraints between the mixture components and/or have computational and sample complexities growing exponentially in the number of mixture components \( r \). In contrast, the so-called spectral methods have both computational and sample complexities scaling only polynomially in the number of components, and do not impose stringent separation constraints, and we outline below.

**Spectral Methods for Mixtures of Product Distributions:** The classical mixture model over product distributions consists of multivariate distributions with a single latent variable \( H \) and the observed variables are conditionally independent under each state of the latent variable (Lazarsfeld and Henry, 1968). Hierarchical latent class (HLC) models (Chen et al., 2008; Zhang, 2004; Zhang and Kocka, 2004) generalize these models by allowing for multiple latent variables. Spectral methods were first employed for learning discrete (hierarchical) mixtures of product distributions (Chang, 1996; Hsu et al., 2009; Mossel and Roch, 2006) and have been recently extended for learning general multiview mixtures (Anandkumar et al., 2012a). The method is based on triplet and pairwise statistics of observed variables and we build on these methods in our work. Note that our setting is not a mixture of product distributions, and thus, these methods are not directly applicable.

**Graphical Model Selection:** Graphical model selection is a well studied problem starting from the seminal work of Chow and Liu (1968) for finding the best tree approximation of a graphical model. They established that maximum likelihood estimation reduces to a maximum weight spanning tree problem where the edge weights are given by empirical mutual information. However, the problem becomes more challenging when either some of the nodes are hidden (i.e., latent tree models) or we are interested in estimating loopy graphs. Learning the structure of latent tree models has been studied extensively, mainly in the context of phylogenetics (Durbin et al., 1999). Efficient algorithms with provable performance guarantees are available, e.g. (Anandkumar et al., 2011; Choi et al., 2011; Daskalakis et al., 2006; Erdős et al., 1999). Works on high-dimensional loopy graphical model selection are more recent. The approaches can be classified into mainly two groups: non-convex local approaches (Anandkumar and Valluvan, 2012; Anandkumar et al., 2012c; Bresler et al., 2008; Jalali et al., 2011; Netrapalli et al., 2010) and those based on convex optimization (Chandrasekaran et al., 2010; Meinshausen and Bühlmann, 2006; Ravikumar et al., 2008, 2011). There is also some recent work on learning conditional models, e.g. (Guo et al., 2011). However, these works are not directly applicable for learning mixtures of graphical models.
Mixtures of Graphical Models: Works on learning mixtures of graphical models (other than mixtures of product distributions) are fewer, and mostly focus on tree mixtures. The works of Meila and Jordan (2001) and Kumar and Koller (2009) consider EM-based approaches for learning tree mixtures. Thiesson et al. (1999) extend the approach to learn mixtures of graphical models on directed acyclic graphs (DAG), termed as Bayesian multinet by Geiger and Heckerman (1996), using the Cheeseman-Stutz asymptotic approximation and Armstrong et al. (2009) consider a Bayesian approach by assigning a prior to decomposable graphs. However, these approaches do not have any theoretical guarantees.

Recently, Mossel and Roch (2011) consider structure learning of latent tree mixtures and provide conditions under which they can be successfully recovered. Note that this model can be thought of as a hierarchical mixture of product distributions, where the hierarchy changes according to the realization of the choice variable. Our setting differs substantially from this work. Mossel and Roch (2011) require that the component latent trees of the mixture be very different, in order for the quartet tests to distinguish them (roughly), and establish that a uniform selection of trees will ensure this condition. On the other hand, we impose no such restriction and allow for graphs of different components to be same/different (although our algorithm is efficient when the overlap between the component graphs is more). Moreover, we allow for loopy graphs while Mossel and Roch (2011) restrict to learning latent tree mixtures. However, Mossel and Roch (2011) do allow for latent variables on the tree, while we assume that all variables to be observed (except for the latent choice variable). Mossel and Roch (2011) consider only structure learning, while we consider both structure and parameter estimations. Mossel and Roch (2011) limit to finite number of mixtures $r = O(1)$, while we allow for $r$ to scale with the number of variables $p$. As such, the two methods operate in significantly different settings.

2 System Model

2.1 Graphical Models

We first introduce the concept of a graphical model and then discuss mixture models. A graphical model is a family of multivariate distributions Markov on a given undirected graph (Lauritzen, 1996). In a discrete graphical model, each node in the graph $v \in V$ is associated with a random variable $Y_v$ taking value in a finite set $\mathcal{Y}$ and let $d := |\mathcal{Y}|$ denote the cardinality of the set. The set of edges $E \subset \binom{V}{2}$ captures the set of conditional-independence relationships among the random variables. We say that a vector of random variables $\mathbf{Y} := (Y_1, \ldots, Y_p)$ with a joint probability mass function (pmf) $P$ is Markov on the graph $G$ if the local Markov property

$$P(y_v | y_{\mathcal{N}(i)}) = P(y_v | y_{\mathcal{V} \setminus v})$$

holds for all nodes $v \in V$, where $\mathcal{N}(v)$ denotes the open neighborhood of $v$ (i.e., not including $v$). More generally, we say that $P$ satisfies the global Markov property for all disjoint sets $A, B \subset V$

$$P(\mathbf{y}_A, \mathbf{y}_B | \mathbf{y}_{S(A,B,G)}) = P(\mathbf{y}_A | \mathbf{y}_{S(A,B,G)}) P(\mathbf{y}_B | \mathbf{y}_{S(A,B,G)}), \quad \forall A, B \subset V : \mathcal{N}[A] \cap \mathcal{N}[B] = \emptyset.$$  

where the set $S(A, B; G)$ is a node separator between $A$ and $B$, and $\mathcal{N}[A]$ denotes the closed neighborhood of $A$ (i.e., including $A$). The global and local Markov properties are equivalent under

$^1$We use notations $E$ and $G$ interchangeably to denote the set of edges.

$^2$A set $\mathcal{S}(A, B; G) \subset V$ is a separator of sets $A$ and $B$ if the removal of nodes in $\mathcal{S}(A, B; G)$ separates $A$ and $B$ into distinct components.
the positivity condition, given by \( P(y) > 0 \), for all \( y \in \mathcal{Y} \) [Lauritzen, 1996]. Henceforth, we say that a graphical model satisfies Markov property with respect to a graph, if it satisfies the global Markov property.

The Hammersley-Clifford theorem [Brémaud, 1999] states that under the positivity condition, a distribution \( P \) satisfies the Markov property according to a graph \( G \) iff. it factorizes according to the cliques of \( G \),

\[
P(y) = \frac{1}{Z} \exp \left( \sum_{c \in \mathcal{C}} \Psi_c(y_c) \right),
\]

where \( \mathcal{C} \) is the set of cliques of \( G \) and \( y_c \) is the set of random variables on clique \( c \). The quantity \( Z \) is known as the partition function and serves to normalize the probability distribution. The functions \( \Psi_c \) are known as potential functions. We will assume positivity of the graphical models under consideration, but otherwise allow for general potentials (including higher order potentials).

### 2.2 Mixtures of Graphical Models

In this paper, we consider mixtures of discrete graphical models. Let \( H \) denote the discrete hidden choice variable corresponding to the selection of a different components of the mixture, taking values in \([r] := \{1, \ldots, r\}\) and let \( Y \) denote the observed variables of the mixture. Denote \( \pi_H := [P(H = h)]_h^\top \) as the probability vector of the mixing weights and \( G_h \) as the Markov graph of the distribution \( P(y|H = h) \).

Our goal is to learn the mixture of graphical models, given \( n \) i.i.d. samples \( y^n := [y_1, \ldots, y_n]^\top \) drawn from a \( p \)-variate joint distribution \( P(y) \) of the mixture model, where each variable is a \( d \)-dimensional discrete variable. The component Markov graphs \( \{G_h\}_h \) corresponding to models \( \{P(y|H = h)\}_h \) are assumed to be unknown. Moreover, the variable \( H \) is latent and thus, we do not a priori know the mixture component from which a sample is drawn. This implies that we cannot directly apply the previous methods designed for graphical model selection. A major challenge is thus being able to decompose the observed statistics into the mixture components.

We now propose a method for learning the mixture components given \( n \) i.i.d. samples \( y^n \) drawn from a graphical mixture model \( P(y) \). Our method proceeds in three main stages. First, we estimate the graph \( G_U := \bigcup_{h=1} G_h \), which is the union of the Markov graphs of the mixture. This is accomplished via a series of rank tests. Note that in the special case when \( G_h \equiv G_U \), this gives the graph estimates of the component models. We then use the graph estimate \( G_U \) to obtain the pairwise marginals of the respective mixture components via a spectral decomposition method. Finally, we use the Chow-Liu algorithm to obtain tree approximations \( \{T_h\}_h \) of the individual mixture components.

### 3 Estimation of the Union of Component Graphs

**Notation:** Our learning method will be based on the estimates of probability matrices. For any two nodes \( u, v \in V \) and any set \( S \subseteq V \setminus \{u, v\} \), denote the joint probability matrix

\[
M_{u,v,S;k} := [P(Y_u = i, Y_v = j, Y_S = k)]_{i,j}, \quad k \in \mathcal{Y}^{|S|}.
\]
Let $\hat{M}^n_{u,v;\{S;k\}}$ denote the corresponding estimated matrices using samples $y^n$.

$$\hat{M}^n_{u,v;\{S;k\}} := [\hat{P}^n(Y_u = i, Y_v = j, Y_S = k)]_{i,j}, \quad (5)$$

where $\hat{P}^n$ denotes the empirical probability distribution, computed using $n$ samples. We consider

sets $S$ satisfying $|S| \leq \eta$, where $\eta$ depends on the graph family under consideration. Thus, our

method is based on $(\eta + 2)^{th}$ order statistics of the observed variables.

**Intuitions:** We provide some intuitions and properties of the union graph $G_{\cup} = \bigcup_{h=1}^{r} G_h$, where

$G_h$ is the Markov graph corresponding to component $H = h$. Note that $G_{\cup}$ is different from the

Markov graph corresponding to the marginalized model $P(y)$ (with latent choice variable $H$ marginalized out). Yet, $G_{\cup}$ represents some natural Markov properties with respect to the observed

statistics. We first establish the simple result that the union graph $G_{\cup}$ satisfies Markov property

in each mixture component. Recall that $S(u,v; G_{\cup})$ denotes a vertex separator between nodes $u$ and $v$ in $G_{\cup}$, i.e., its removal disconnects $u$ and $v$ in $G_{\cup}$.

**Fact 1 (Markov Property of $G_{\cup}$)** For any two nodes $u, v \in V$ such that $(u, v) \notin G_{\cup}$,

$$Y_u \perp Y_v | Y_S, H, \quad S := S(u,v; G_{\cup}). \quad (6)$$

**Proof:** The set $S := S(u,v; G_{\cup})$ is also a vertex separator for $u$ and $v$ in each of the component

graphs $G_h$. This is because removal of $S$ disconnects $u$ and $v$ in each $G_h$. Thus, we have Markov property in each component: $Y_u \perp Y_v | Y_S, \{H = h\}$, for $h \in [r]$, and the above result follows. □

Thus, the above observation implies that the conditional independence relationships of each mixture component are satisfied on the union graph $G_{\cup}$ conditioned on the latent factor $H$. The above result can be exploited to obtain union graph estimate as follows: two nodes $u, v$ are not neighbors in $G_{\cup}$ if a separator set $S$ can be found which results in conditional independence, as in (6). The main challenge is indeed that the variable $H$ is not observed and thus, conditional independence cannot be directly inferred via observed statistics. However, the effect of $H$ on the observed statistics can be quantified as follows:

**Lemma 1 (Rank Property)** Given an $r$-component mixture of graphical models with $G_{\cup} = \bigcup_{h=1}^{r} G_h$, for any $u,v \in V$ such that $(u,v) \notin G_{\cup}$ and $S := S(u,v; G_{\cup})$, the probability matrix

$M_{u,v;\{S;k\}} := [P(y_u = i, y_v = j, y_S = k)]_{i,j}$ has rank at most $r$ for any $k \in Y^{[S]}$.

**Proof:** From Fact [1] $G_{\cup}$ satisfies Markov property conditioned on the latent factor $H$,

$$Y_u \perp Y_v | Y_S, H, \quad \forall (u,v) \notin G_{\cup}. \quad (7)$$

This implies that

$$M_{u,v;\{S;k\}} = M_{u|H;\{S;k\}} \text{Diag}(\pi_{H|\{S;k\}}) M_{v|H;\{S;k\}}^T P(Y_S = k), \quad (8)$$

where $M_{u|H;\{S;k\}} := [P(y_u = i|H = j, Y_S = k)]_{i,j}$ and similarly $M_{v|H;\{S;k\}}$ is defined. $\text{Diag}(\pi_{H|\{S;k\}})$ is the diagonal matrix with entries $\pi_{H|\{S;k\}} := [P(H = i|Y_S = k)]_{i}$. Thus, we have $\text{Rank}(M_{u,v;\{S;k\}})$ is at most $r$.

Thus, the effect of marginalizing the choice variable $H$ is seen in the rank of the observed probability matrices $M_{u,v;\{S;k\}}$. Thus, when $u$ and $v$ are non-neighbors in $G_{\cup}$, a separator set $S$
can be found such that the rank of \( M_{u,v}\{S;k\} \) is at most \( r \). In order to use this result as a criterion for inferring neighbors in \( G_{\cup} \), we require that the rank of \( M_{u,v}\{S;k\} \) for any neighbors \((u,v) \in G_{\cup}\) be strictly larger than \( r \). This requires the dimension of each node variable \( d > r \). We discuss in detail the set of sufficient conditions for correctly recovering \( G_{\cup} \) in Section 3.1.1.

**Tractable Graph Families:** Another obstacle in using Lemma 1 to estimate graph \( G_{\cup} \) is computational: the search for separators \( S \) for any node pair \( u,v \in V \) is exponential in \( |V| := p \) if no further constraints are imposed. We consider graph families where a vertex separator can be found for any \((u,v) \notin G_{\cup}\) with size at most \( \eta \)

\[
|S(u,v; G_{\cup})| \leq \eta, \quad \forall (u,v) \notin G_{\cup}.
\]

There are many natural families where \( \eta \) is small:

1. If \( G_{\cup} \) is trivial (i.e., no edges) then \( \eta = 0 \), we have a mixture of product distributions.
2. When \( G_{\cup} \) is a tree, i.e., we have a mixture model Markov on the same tree, then \( \eta = 1 \), since there is a unique path between any two nodes on a tree.
3. For an arbitrary \( r \)-component tree mixture, \( G_{\cup} = \bigcup_h T_h \) where each component is a tree distribution, we have that \( \eta \leq r \) (since for any node pair, there is a unique path in each of the \( r \) trees \( \{T_h\} \), and separating the node pair in each \( T_h \) also separates them on \( G_{\cup} \)).
4. For an arbitrary mixture of bounded degree graphs, we have \( \eta \leq \sum_{h \in [r]} \Delta_h \), where \( \Delta_h \) is the maximum degree in \( G_h \), i.e., the Markov graph corresponding to component \( \{H = h\} \).

In general, \( \eta \) depends on the respective bounds \( \eta_h \) for the component graphs \( G_h \), as well as the extent of their overlap. In the worst case, \( \eta \) can be as high as \( \sum_{h \in [r]} \eta_h \), while in the special case when \( G_h \equiv G_{\cup} \), the bound remains the same \( \eta_h \equiv \eta \). Note that for a general graph \( G_{\cup} \) with tree-width \( \text{tw}(G_{\cup}) \) and maximum degree \( \Delta(G_{\cup}) \), we have that \( \eta \leq \min(\Delta(G_{\cup}), \text{tw}(G_{\cup})) \). Thus, a wide family of models give rise to union graph with small \( \eta \), including tree mixtures and mixtures over bounded degree graphs.

We establish in the sequel that the computational and sample complexities of our learning method scale exponentially in \( \eta \). Thus, our algorithm is suitable for graphs \( G_{\cup} \) with small \( \eta \). In Section 3 we relax the requirement of exact separation to that of local separation. A larger class of graphs satisfy the local separation property including mixtures of locally tree-like graphs.

**Rank Test:** We propose \( \text{RankTest}(y^n; \xi_{n,p}, \eta, r) \) in Algorithm 1 for structure estimation of \( G_{\cup} := \bigcup_{h=1}^r G_h \), the union Markov graph of an \( r \)-component mixture. The method is based on a search for potential separators \( S \) between any two given nodes \( u,v \in V \), based on the effective rank \( \hat{r} \) of \( \hat{M}_{u,v;\{S;k\}} ^n \): if the effective rank is \( r \) or less, then \( u \) and \( v \) are declared as non-neighbors (and set \( S \) as their separator). If no such sets are found, they are declared as neighbors. Thus, the method involves searching for separators for each node pair \( u,v \in V \), by considering all sets \( S \subset V \setminus \{u,v\} \) satisfying \(|S| \leq \eta \). The computational complexity of this procedure is \( O(p^{r+2}d^2) \), where \( d \) is the dimension of each node variable \( Y_i \), for \( i \in V \) and \( p \) is the number of nodes. This is because the number of rank tests performed is \( O(p^{r+2}) \) over all node pairs and conditioning sets; each rank test has \( O(d^2) \) complexity since it involves performing singular value decomposition (SVD) of a \( d \times d \) matrix.

\[ \text{The effective rank is given by the number of singular values above a given threshold } \xi. \]
Algorithm 1 $\hat{G}_n^\cup = \text{RankTest}(y^n; \xi_{n,p}, \eta, r)$ for estimating $G_\cup := \bigcup_{h=1}^r G_h$ of an $r$-component mixture using $y^n$ samples, where $\eta$ is the bound on size of vertex separators between any node pair in $G_\cup$ and $\xi_{n,p}$ is a threshold on the singular values.

$\text{Rank}(A; \xi)$ denotes the effective rank of matrix $A$, i.e., number of singular values more than $\xi$. $\hat{M}^n_{u,v,\{S;k\}} := [\hat{P}^n(Y_u = i, Y_v = j, Y_S = k)]_{i,j}$ is the empirical estimate computed using $n$ i.i.d. samples $y^n$. Initialize $\hat{G}_n^\cup = (V, \emptyset)$.

For each $u, v \in V$, estimate $\hat{M}^n_{u,v,\{S;k\}}$ from $y^n$, if
\[
\min_{S \subset V \setminus \{u,v\}} \max_{k \in \mathcal{Y}^{|S|}} \text{Rank}(\hat{M}^n_{u,v,\{S;k\}}; \xi_{n,p}) > r,
\]
then add $(u, v)$ to $\hat{G}_n^\cup$.

3.1 Results for the Rank Test

3.1.1 Conditions for the Success of Rank Tests

The following assumptions are made for the RankTest proposed in Algorithm 1 to succeed under the PAC formulation.

(A1) **Number of Mixture Components**: The number of components $r$ of the mixture model and dimension $d$ of each node variable satisfy
\[ d > r. \]

The mixing weights of the latent factor $H$ are assumed to be strictly positive
\[ \pi_H(h) := P(H = h) > 0, \quad \forall h \in [r]. \]

(A2) **Constraints on Graph Structure**: Recall that $G_\cup = \bigcup_{h=1}^r G_h$ denotes the union of the graphs of the components and that $\eta$ denotes the bound on the size of the minimal separator set for any two (non-neighboring) nodes in $G_\cup$. We assume that
\[ |S(u, v; G_\cup)| \leq \eta = O(1), \quad \forall (u, v) \notin G_\cup. \]

In Section B, we relax the strict separation constraint to a local separation constraint in the regime of correlation decay, where $\eta$ refers to the bound on the size of local separators between any two non-neighbor nodes in the graph.

(A3) **Rank Condition**: We assume that the matrix $M_{u,v,\{S;k\}}$ in (4) has rank strictly greater than $r$ when the nodes $u$ and $v$ are neighbors in graph $G_\cup = \bigcup_{h=1}^r G_h$ and the set satisfies $|S| \leq \eta$. Let $\rho_{\min}$ denote
\[ \rho_{\min} := \min_{(u, v) \in G_\cup, |S| \leq \eta} \max_{k \in \mathcal{Y}^{|S|}} \sigma_{r+1}(M_{u,v,\{S;k\}}) > 0, \]
where $\sigma_{r+1}(\cdot)$ denotes the $(r + 1)^{th}$ singular value, when the singular values are arranged in the descending order $\sigma_1(\cdot) \geq \sigma_2(\cdot) \geq \ldots \sigma_d(\cdot)$. 
(A4) **Choice of threshold** $\xi$: For $\text{RankTest}$ in Algorithm 1, the threshold $\xi$ is chosen as

$$\xi := \frac{\rho_{\min}}{2}.$$ 

(A5) **Number of Samples:** Given $\delta \in (0, 1)$, the number of samples $n$ satisfies

$$n > n_{\text{Rank}}(\delta; p) := \max \left( \frac{1}{12} \left( 2 \log p + \log \delta^{-1} + \log 2 \right), \left( \frac{2}{\rho_{\min} - t} \right)^2 \right),$$

for some $t \in (0, \rho_{\min})$ (e.g. $t = \rho_{\min}/2$) where $p$ is the number of nodes and $\rho_{\min}$ is given by (11).

Assumption (A1) relates the number of components to the dimension of the sample space of the variables. Note that we allow for the number of components $r$ to grow with the number of nodes $p$, as long as the cardinality of the sample space of each variable $d$ is also large enough. In principle, this assumption can be removed by considering grouping the nodes together and performing rank tests on the groups. Assumption (A2) imposes constraints on the graph structure $G_{\cup}$, formed by the union of the component graphs. The bound $\eta$ on the separator sets for node pairs in $G_{\cup}$ is a crucial parameter and the complexity of learning (both sample and computational) depends on it. We relax the assumption of separator bound to a criterion of local separation in Section [B]. Assumption (A3) is required for the success of rank tests to distinguish neighbors and non-neighbors in graph $G_{\cup}$. It rules out the presence of spurious low rank matrices between neighboring nodes in $G_{\cup}$ (for instance, when the nodes are marginally independent or when the distribution is degenerate). Assumption (A4) provides a natural threshold on the singular values in the rank test. In Section [B], we modify the threshold to also account for distortion due to approximate vertex separation, in contrast to the setting of exact separation considered in this section. (A5) provides the finite sample complexity bound.

### 3.1.2 Result on Rank Tests

We now provide the result on the success of recovering the graph $G_{\cup} := \bigcup_{h=1}^{r} G_h$.

**Theorem 1 (Success of Rank Tests)** 

The $\text{RankTest}(y^n; \xi, \eta, r)$ outputs the correct graph $G_{\cup} := \bigcup_{h=1}^{r} G_h$, which is the union of the component Markov graphs, under the assumptions (A1)–(A5) with probability at least $1 - \delta$.

**Proof:** The proof is given in Appendix [C] \hfill \Box

A special case of the above result is graphical model selection, where there is a single graphical model ($r = 1$) and we are interested in estimating its graph structure.

**Corollary 1 (Application to Graphical Model Selection)** 

The $\text{RankTest}(y^n; \xi, \eta, 1)$ outputs the correct Markov graph $G$, given $n$ i.i.d. samples $y^n$, under the assumptions (A2)–(A5) with probability at least $1 - \delta$.

---

5When $r = 1$, there is no latent factor, and the assumption $d > r$ in (A1) is trivially satisfied for all discrete random variables.
Remarks: Thus, the rank test is also applicable for graphical model selection. Previous works (see Section 1.2) have proposed tests based on conditional independence, using either conditional mutual information or conditional variation distances, see Anandkumar et al. (2012c); Bresler et al. (2008). The rank test above is thus an alternative test for conditional independence. In addition, it extends naturally to estimation of union graph structure of mixture components.

4 Parameter Estimation of Mixture Components

The rank test proposed in the previous section is a tractable procedure for estimating the graph $G_\cup := \bigcup_{h=1}^r G_h$, which is the union of the component graphs of a mixture of graphical models. However, except in the special case when $G_h \equiv G_\cup$, the knowledge of $\hat{G}_n \cup$ is not very useful by itself, since we do not know the nature of the different components of the mixture. In this section, we propose the use of spectral decomposition tests to find the various mixture components.

4.0.3 Spectral Decomposition for Mixture of Product Distributions

The spectral decomposition methods, first proposed by Chang (1996), and later generalized by Mossel and Roch (2006) and Hsu et al. (2009), and recently by Anandkumar et al. (2012a), are applicable for mixtures of product distributions. We illustrate the method below via a simple example.

Consider the simple case of three observed variables $\{Y_u, Y_v, Y_w\}$, where a latent factor $H$ separates them, i.e., the observed variables are conditionally independent given $H$:

$$Y_u \perp \perp Y_v \perp \perp Y_w | H.$$  

This implies that the Markov graphs $\{G_h\}_{h \in [r]}$ of the component models $\{P(Y_u, Y_v, Y_w | H = h)\}_{h \in [r]}$ are trivial (i.e., have no edges) and thus forms a special case of our setting.

We now give an overview of the spectral decomposition method. It proceeds by considering pairwise and triplet statistics of $Y_u, Y_v, Y_w$. Denote $M_u|H := [P(Y_u = i | H = j)]_{i,j}$, and similarly for $M_v|H, M_w|H$ and assume that they are full rank. Denote the probability matrices $M_{u,v} := [P(Y_u = i, Y_v = j)]_{i,j}$ and $M_{u,v,\{w;k\}} := [P(Y_u = i, Y_v = j, Y_w = k)]_{i,j}$. The parameters (i.e., matrices $M_{u|H}, M_{v|H}, M_{w|H}$) can be estimated as:

Lemma 2 (Mixture of Product Distributions) For the latent variable model $Y_u \perp \perp Y_v \perp \perp Y_w | H$, when the conditional probability matrices $M_{u|H}, M_{v|H}, M_{w|H}$ have rank $d$, let $\lambda^{(k)} = [\lambda_1^{(k)}, \ldots, \lambda_d^{(k)}]^\top$ be the column vector with the $d$ eigenvalues given by

$$\lambda^{(k)} := \text{Eigenvalues} \left( M_{u,v,\{w;k\}} M_{u,v}^{-1} \right), \quad k \in \mathcal{Y}.$$  

Let $\Lambda := [\lambda^{(1)} | \lambda^{(2)} | \ldots | \lambda^{(d)}]$ be the matrix where the $k^{th}$ column corresponds to $\lambda^{(k)}$ from above. We have that

$$M_{w|H} := [P(Y_w = i | H = j)]_{i,j} = \Lambda \Lambda.$$  

Proof: A more general result is proven in Appendix D.1. □

Thus, we have a procedure for recovering the conditional probabilities of the observed variables conditioned on the latent factor. Using these parameters, we can also recover the mixing weights...
\[ \pi_H := [P(H = i)]^T \] using the relationship

\[ M_{u,v} = M_{u|H} \text{Diag}(\pi_H)M_{v|H}^T, \]

where \( \text{Diag}(\pi_H) \) is the diagonal matrix with \( \pi_H \) as the diagonal elements.

Thus, if we have a general product distribution mixture over nodes in \( V \), we can learn the parameters by performing the above spectral decomposition over different triplets \( \{u, v, w\} \). However, an obstacle remains: spectral decomposition over different triplets \( \{u, v, w\} \) results in different permutations of the labels of the hidden variable \( H \). To overcome this, note that any two triplets \( (u, v, w) \) and \( (u, v', w') \) share the same set of eigenvectors in (13) when the “left” node \( u \) is the same. Thus, if we consider a fixed node \( u^* \in V \) as the “left” node and use a fixed matrix to diagonalize (13) for all triplets, we obtain a consistent ordering of the hidden labels over all triplet decompositions. Thus, we can learn a general product distribution mixture using only third-order statistics.

### 4.0.4 Spectral Decomposition for Learning Graphical Model Mixtures

We now adapt the above method for learning more general graphical model mixtures. We first make a simple observation on how to obtain mixtures of product distributions by considering separators on the union graph \( G \cup \). For any three nodes \( u, v, w \in V \), which are not neighbors on \( G \cup \), let \( S_{uvw} \) denote a multiway vertex separator, i.e., the removal of nodes in \( S_{uvw} \) disconnects \( u, v \) and \( w \) in \( G \cup \). On lines of Fact [1]

\[ Y_u \perp \perp Y_v \perp \perp Y_w | Y_{S_{uvw}}, H, \quad \forall u, v, w : (u, v), (v, w), (w, u) \notin G \cup \].

(15)

Thus, by fixing the configuration of nodes in \( S_{uvw} \), we obtain a product distribution mixture over \( \{u, v, w\} \). If the previously proposed rank test is successful in estimating \( G \cup \), then we possess correct knowledge of the separators \( S_{uvw} \). In this case, we can obtain estimates \( \{P(Y_w | Y_{S_{uvw}} = k, H = h)\}_h \) by fixing the nodes in \( S_{uvw} \) to \( k \) and using the spectral decomposition described in Lemma [2] and the procedure can be repeated over different triplets \( \{u, v, w\} \). See Fig[1]

Figure 1: By conditioning on the separator set \( S \) on the union graph \( G \cup \), we have a mixture of product distribution with respect to nodes \( \{u, v, w\} \), i.e., \( Y_u \perp \perp Y_v \perp \perp Y_w | Y_S, H \).

An obstacle remains, viz., the permutation of hidden labels over different triplet decompositions \( \{u, v, w\} \). In case of product distribution mixture, as discussed previously, this is resolved by fixing the “left” node in the triplet to some \( u^* \in V \) and using the same matrix for diagonalization over different triplets. However, an additional complication arises when we consider graphical model mixtures, where conditioning over separators is required. We require that the permutation of the
hidden labels be unchanged upon conditioning over different values of variables in the separator set $S_{u,vw}$. This holds when the separator set $S_{u,vw}$ has no effect on node $u_*$, i.e., we require that

$$\exists u_* \in V, s.t. \ Y_{u_*} \perp \perp Y_{V \setminus u_*} | H,$$

which implies that $u_*$ is isolated from all other nodes in graph $G_U$.

Condition (16) is required to hold for identifiability if we only operate on statistics over different triplets (along with their separator sets). In other words, if we resort to operations over only low order statistics, we require additional conditions such as (16) for identifiability. However, our setting is a significant generalization over the mixtures of product distributions, where (16) is required to hold for all nodes.

Finally, since our goal is to estimate pairwise marginals of the mixture components, in place of node $w$ in the triplet $\{u, v, w\}$ in Lemma 2, we need to consider a node pair $a, b \in V$. The general algorithm allows the variables in the triplet to have different dimensions, see Anandkumar et al. (2012a) for details. Thus, we obtain estimates of the pairwise marginals of the mixture components.

The computational complexity of the procedure scales as $O(p^2d^\eta + 6r)$, where $p$ is the number of nodes, $d$ is the cardinality of each node variable and $\eta$ is the bound on separator sets. For details on implementation of the spectral method, see Appendix A.

### 4.1 Results for Spectral Decomposition

#### 4.1.1 Assumptions

In addition to the assumptions (A1)–(A5) in Section 3.1.1, we impose the following constraints to guarantee the success of estimating the various mixture components.

(A6) **Full Rank Views of the Latent Factor:** For each node pair $a, b \in V$, and any subset $S \subset V \setminus \{a, b\}$ with $|S| \leq 2\eta$ and $k \in \mathcal{Y}^{|S|}$, the probability matrix $M_{(a,b)|H,\{S;k\}} := [P(Y_{a,b} = i| H = j, Y_S = k)]_{i,j} \in \mathbb{R}^{d^2 \times r}$ has rank $r$.

(A7) **Existence of an Isolated Node:** There exists a node $u_* \in V$ which is isolated from all other nodes in $G_U = \bigcup_{h=1}^r G_h$, i.e.

$$Y_{u_*} \perp \perp Y_{V \setminus u_*} | H.$$

(A8) **Spectral Bounds and Random Rotation Matrix:** Refer to various spectral bounds used to obtain $K(\delta; p, d, r)$ in Appendix [D.3] where $\delta \in (0,1)$ is fixed. Further assume that the rotation matrix $Z \in \mathbb{R}^{r \times r}$ in FindMixtureComponents is chosen uniformly over the Stiefel manifold $\{Q \in \mathbb{R}^{r \times r} : Q^\top Q = I\}$.

(A9) **Number of Samples:** For fixed $\delta, \epsilon \in (0,1)$, the number of samples satisfies

$$n > n_{\text{spect}}(\delta; \epsilon; p, d, r) := \frac{4K^2(\delta; p, d, r)}{\epsilon^2},$$

where $K(\delta; p, d, r)$ is defined in (58).

Assumption (A6) is a natural condition required for the success of spectral decomposition, and is imposed in (Mossel and Roch, 2006), (Hsu et al., 2009) and (Anandkumar et al., 2012a). It is also known that learning singular models, i.e., those which do not satisfy (A6), is at least as hard
as learning parity with noise, which is conjectured to be computationally hard (Mossel and Roch, 2006). The condition in (A7) is indeed an additional constraint on graph $G_\cup$, but is required to ensure alignment of hidden labels over spectral decompositions of different groups of variables, as discussed before. Condition (A8) assumes various spectral bounds and (A9) characterizes the sample complexity.

4.1.2 Guarantees for Learning Mixture Components

We now provide the result on the success of recovering the tree approximation $T_h$ of each mixture component $P(y|H = h)$. Let $\|\cdot\|_2$ on a vector denote the $\ell_2$ norm.

**Theorem 2 (Guarantees for FindMixtureComponents)** Under the assumptions (A1)–(A9), the procedure in Algorithm 2 outputs $\hat{P}^{\text{spect}}(Y_a, Y_b|H = h)$, for each $a, b \in V$, such that for all $h \in [r]$, there exists a permutation $\tau(h) \in [r]$ with

$$\|\hat{P}^{\text{spect}}(Y_a, Y_b|H = h) - P(Y_a, Y_b|H = \tau(h))\|_2 \leq \epsilon,$$

(19)

with probability at least $1 - 4\delta$.

**Proof:** The proof is given in Appendix D. $\blacksquare$

**Remarks:** Recall that $p$ denotes the number of variables, $r$ denotes the number of mixture components, $d$ denotes the dimension of each node variable and $\eta$ denotes the bound on separator sets between any node pair in the union graph. The quantity $K(\delta; p, d, r)$ in (58) in Appendix D.3 is $O(p^{2\eta + 2d^2\eta + 5}\delta^{-1}\log(p, d, r, \delta^{-1}))$. Thus, we require the number of samples scaling in (18) as $n = \Omega(p^{4\eta + 4d^2\eta + 10}\delta^{-2}\epsilon^{-2}\log(p, d, r, \delta^{-1}))$. Since we operate in the regime where $\eta = O(1)$ is a small constant, this implies that we have a polynomial sample complexity in $p, d, r$. Note that the special case of $\eta = 0$ corresponds to the case of mixture of product distributions, and it has the best sample complexity.

4.1.3 Analysis of Tree Approximation

We now consider the final stage of our approach, viz., learning tree approximations using the estimates of the pairwise marginals of the mixture components from the spectral decomposition method. We now impose a standard condition of non-degeneracy on each mixture component to guarantee the existence of a unique tree structure corresponding to the maximum-likelihood tree approximation to the mixture component.

**(A10) Separation of Mutual Information:** Let $T_h$ denote the Chow-Liu tree corresponding to the model $P(y|H = h)$ when exact statistics are input and let

$$\vartheta := \min_{h \in [r]} \min_{(a,b) \notin T_h} \min_{(u,v) \in \text{Path}(a,b; T_h)} (I(Y_u, Y_v|H = h) - I(Y_a, Y_b|H = h)),$$

(20)

where $\text{Path}(a,b; T_h)$ denotes the edges along the path connecting $a$ and $b$ in $T_h$.

---

(A7) can be relaxed as follows: if graph $G_\cup$ has at least three connected components, then we can choose a reference node in each of the components and estimate the marginals in the other components. For instance, if $C_1, C_2, C_3$ are three connected components in $G_\cup$, then we can choose a node in $C_1$ as the reference node to estimate the marginals of $C_2$ and $C_3$. Similarly, we can choose a node in $C_2$ as a reference node and estimate the marginals in $C_1$ and $C_3$. We can then align these different estimates and obtain all the marginals.
(A11) **Number of Samples:** For $\epsilon_{\text{tree}}$ defined in (69), the number of samples is now required to satisfy

$$n > n_{\text{spect}}(\delta; \epsilon_{\text{tree}}; p, d, r),$$

where $n_{\text{spect}}$ is given by (18).

The condition in (A10) assumes a separation between mutual information along edges and non-edges of the Chow-Liu tree $T_h$ of each component model $P(y|H = h)$. The quantity $\vartheta$ represents the minimum separation between the mutual information along an edge and any set of non-edges which can replace the edge in $T_h$. Note that $\vartheta \geq 0$ due to the max-weight spanning tree property of $T_h$ (under exact statistics). Intuitively $\vartheta$ denotes the “bottleneck” where errors are most likely to occur in tree structure estimation. Similar observations were made by [Tan et al. (2011)] for error exponent analysis of Chow-Liu algorithm. The sample complexity for correctly estimating $T_h$ using samples is based on $\vartheta_h$ and given in (A11). This ensures that the mutual information quantities are estimated within the separation bound $\vartheta$.

**Theorem 3 (Tree Approximations of Mixture Components)** Under (A1)–(A11), the Chow-Liu algorithm outputs the correct tree structures corresponding to maximum-likelihood tree approximations of the mixture components $\{P(y|H = h)\}$ with probability at least $1 - 4\delta$, when the estimates of pairwise marginals $\{P_{\text{spect}}(Y_a, Y_b|H = h)\}$ from spectral decomposition method are input.

*Proof:* See Section [D.5] □

**Remarks:** Thus our approach succeeds in recovering the correct tree structures corresponding to ML-tree approximations of mixture components with computational and sample complexities scaling polynomially in the number of variables $p$, number of components $r$ and the dimension of each variable $d$.

Note that if the underlying model is a tree mixture, we recover the tree structures of the mixture components. For this special case, we can give a slightly better guarantee by estimating Chow-Liu trees which are subgraphs of the union graph estimate $\tilde{G}_\cup$, and this is discussed in Appendix [D.3]. The improved bound $K_{\text{tree}}(\delta; p, d, r)$ is $O\left(p^2(d\Delta)^{2n}r^5\delta^{-1}\text{poly log}(p, d, r, \delta^{-1})\right)$, where $\Delta$ is the maximum degree in $G_\cup$.

5 Conclusion

In this paper, we considered learning tree approximations of graphical model mixtures. We proposed novel methods which combined techniques used previously in graphical model selection, and in learning mixtures of product distributions. We provided provable guarantees for our method, and established that it has polynomial sample and computational complexities in the number of nodes $p$, number of mixture components $r$ and cardinality of each node variable $d$. Our guarantees are applicable for a wide family of models. In future, we plan to investigate learning mixtures of continuous models, such as Gaussian mixture models.

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A Implementation of Spectral Decomposition Method

Overview of the algorithm: We provide the procedure in Algorithm 2. The algorithm computes the pairwise statistic of each node pair \(a, b \in V \setminus \{u_\ast\}\), where \(u_\ast\) is the reference node which is isolated in \(\hat{G}_J\), the union graph estimate obtained using Algorithm 1. The spectral decomposition is carried out on the triplet \(\{u_\ast, c, (a,b); \{S = k\}\}\), where \(c\) is any node not in the neighborhood of \(a\) or \(b\) in graph \(\hat{G}_J\). Set \(S \subset V \setminus \{a, b, u_\ast\}\) is separates \(a, b\) from \(c\) in \(\hat{G}_J\). See Fig. 2. We fix the configuration of the separator set to \(Y_S = k\), for each \(k \in \mathcal{Y}^{\mid S\mid}\), and consider the empirical distribution of \(n\) samples, \(\hat{P}_n(Y_{u_\ast}, Y_a, Y_b, Y_c; \{Y_S = k\})\). Upon spectral decomposition, we obtain the mixture components \(\hat{\pi}_n^{\text{spect}}(Y_a, Y_b, Y_S | H = h)\) for \(h \in [r]\). We can then employ the estimated pairwise marginals to find the Chow-Liu tree approximation \(\{\hat{T}_h\}_h\) for each mixture component. This routine can also be adapted to estimate the individual Markov graphs \(\{G_h\}_h\) and is described briefly in Section A.1. Also, if the underlying model is a tree mixture, we can slightly modify the algorithm and obtain better guarantees, and we outline it in Section A.1.

Fig. 2: By conditioning on the separator set \(S\) on the union graph \(G_J\), we have a mixture of product distribution with respect to nodes \(\{u_\ast, c, (a,b)\}\), i.e., \(Y_{u_\ast} \perp \perp Y_c \perp \perp Y_{a,b} | Y_S, H\).

Algorithm 2 FindMixtureComponents\((y^n, \hat{G}; r)\) for finding the tree-approximations of the components \(\{P(y|H = h)\}_h\) of an \(r\)-component mixture using samples \(y^n\) and graph \(\hat{G}\), which is an estimate of the graph \(G_J := \bigcup_{h=1}^r G_h\) obtained using Algorithm 1.

\[
\hat{M}_{A,B;\{C;k\}}^n := [P(Y_A = i, Y_B = j, Y_C = k)]_{i,j}\text{ denotes the empirical joint probability matrix estimated using samples } y^n, \text{ where } A \cap B \cap C = \emptyset. \text{ Let } S(A, B; G_J) \text{ be a minimal vertex separator separating } A \text{ and } B \text{ in graph } \hat{G}_J. \text{ Choose a uniformly random orthonormal basis } \{z_1, \ldots, z_r\} \in \mathbb{R}^r. \text{ Let } Z \in \mathbb{R}^{r \times r} \text{ be a matrix whose } l^{th} \text{ row is } z_l^T. \text{ Let } u_\ast \in V \text{ be isolated from all the other nodes in graph } \hat{G}. \text{ Otherwise declare fail.}
\]

for \(a, b \in V \setminus \{u_\ast\}\) do
- Let \(c \notin \mathcal{N}(a; \hat{G}) \cup \mathcal{N}(b; \hat{G})\) (if no such node is found, go to the next node pair). \(S \leftarrow S((a, b); c; \hat{G}).\)
- \(\{\hat{\pi}_n^{\text{spect}}(Y_a, Y_b, Y_S | H = h)\}_h \leftarrow \text{SpecDecom}(u_\ast, c, (a,b); S, y^n, r, Z).\)
end for
for \(h \in [r]\) do
- \([\hat{T}_h, \{\hat{\pi}_n^{\text{tree}}(Y_a, Y_b | H = h)\}_{(a,b) \in \hat{T}_h}] \leftarrow \text{ChowLiuTree}\left(\{\hat{\pi}_n^{\text{spect}}(Y_a, Y_b | H = h)\}_{a,b \in V \setminus \{u_\ast\}}\right).\)
end for
Output \([\hat{\pi}_n^{\text{spect}}(h), \hat{T}_h, \{\hat{\pi}_n^{\text{tree}}(Y_a, Y_b | H = h) : (a,b) \in \hat{T}_h\}]_{h \in [r]}\).
Procedure 3 \[\{\hat{P}(Y_w, Y_S|H = h), \hat{\pi}_H(h)\}_h\leftarrow \text{SpecDecom}(u, v; S, y^n, r, Z)\] for finding the components of an \(r\)-component mixture from \(y^n\) samples at \(w\), given witnesses \(u, v\) and separator \(S\) on graph \(G^n\).

Let \(\hat{M}^n_{w,v;\{S;k\}} := [\hat{P}^n(Y_u = i, Y_v = j, Y_S = k)]_{i,j}\) where \(\hat{P}^n\) is the empirical distribution computed using samples \(y^n\). Similarly, let \(\hat{M}^n_{u,v;\{S;k\},\{w;l\}} := [\hat{P}^n(Y_u = i, Y_v = j, Y_S = k, Y_w = l)]_{i,j}\) for a vector \(\lambda\), let \(\text{Diag}(\lambda)\) denote the corresponding diagonal matrix.

for \(k \in \mathcal{Y}^{|S|}\) do
  Choose \(U_w\) as the set of top \(r\) left orthonormal singular vectors of \(\hat{M}^n_{u,v;\{S;k\}}\) and \(V_v\) as the right singular vectors. Similarly for node \(w\), let \(U_w\) be the top \(r\) left orthonormal singular vectors of \(\hat{M}^n_{w,u;\{S;k\}}\).
  for \(l \in [r]\) do
    \(m_l \leftarrow U_w z_l\), \(A \leftarrow U_u \hat{M}^n_{u,v;\{S;k\}} V_v\) and \(B_l \leftarrow U_u (\sum_q m_l(q) \hat{M}^n_{u,v;\{S;k\},\{w;q\}}) V_v\).
    if \(A\) is invertible (Fail Otherwise) then
      \(C_l \leftarrow B_l A^{-1}\). \(\text{Diag}(\lambda^{(l)}) \leftarrow R^{-1} C_l R\). \{Find \(R\) which diagonalizes \(C_l\) for the first triplet.
      Use the same matrix \(R\) for all other triplets.\}
  end if
end for

Form the matrix from the above eigenvalue computations: \(\Lambda = [\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(r)}]\)

Obtain \(\hat{M}_{w|H,\{S;k\}} \leftarrow U_w Z^{-1} \Lambda^\top\). Similarly obtain \(\hat{M}_{v|H,\{S;k\}}\).

Obtain \(\hat{\pi}_H: \hat{M}^n_{w,v;\{S;k\}} = \hat{M}_{w|H,\{S;k\}} \text{Diag}(\hat{\pi}_H\{S;k\}) (\hat{M}_{v|H,\{S;k\}})^\top \hat{P}^n(Y_S = k)\).

end for

Output \(\{\hat{P}(Y_w, Y_S|H = h), \hat{\pi}_H(h)\}_{h \in [r]}\).

A.1 Discussion and Extensions

Simplification for Tree Mixtures \((G_h = T_h)\): We can simplify the above method by limiting to tree approximations which are subgraphs of graph \(G\). This procedure coincides with the original method when all the component Markov graphs \(\{G_h\}_h\) are trees, i.e., \(G_h = T_h\), \(h \in [r]\). This is because in this case, the Chow-Liu tree coincides with \(T_h \subseteq G\) (under exact statistics). This implies that we need to compute pairwise marginals only over the edges of \(G\) using \text{SpecDecom} routine, instead of over all the node pairs, and the \text{ChowLiuTree} procedure computes a maximum weighted spanning tree over \(G\), instead of the complete graph. This leads a slight improvement of sample complexity, and we note it in the remarks after Theorem 2.

Estimation of Component Markov Graphs \(\{G_h\}_h\): We now note that we can also estimate the component Markov graphs \(\{G_h\}_h\) using the spectral decomposition routines and we briefly describe it below. Roughly, we can do a suitable conditional independence test on the estimated statistics \(\hat{P}^\text{spect}(Y_{\mathcal{N}[a,G]}|H = h)\) obtained from spectral decomposition, for each node neighborhood \(\mathcal{N}[a,G]\), where \(a \in V \setminus \{u_0\}\) and \(G\) is an estimate of \(G := \cup_{h \in [r]} G_h\). We can estimate these statistics by selecting a suitable set of witnesses \(\mathcal{C} := \{c_1, c_2, \ldots\}\) such that \(\mathcal{N}[a]\) can be separated from \(\mathcal{C}\) in \(G\). We can employ Procedure \text{SpecDecom} on this configuration by using a suitable separator set and then doing a threshold test on the estimated component statistics \(\hat{P}^\text{spect}:\)

17
**Procedure 4** $[\hat{T}, \{\hat{P}^{\text{tree}}(Y_a, Y_b)\}_{(a,b) \in \hat{T}}] \leftarrow \text{ChowLiuTree}(\{\hat{P}(Y_a, Y_b)\}_{a,b \in V \setminus \{u_\ast\}})$ for finding a tree approximation given the pairwise statistics.

| for $a,b \in V \setminus \{u_\ast\}$ do |
| Compute mutual information $\hat{I}(Y_a; Y_b)$ using $\hat{P}(Y_a, Y_b)$. |
| end for |
| $\hat{T} \leftarrow \text{MaxWtTree}(\{\hat{I}(Y_a; Y_b)\})$ is max-weight spanning tree using edge weights $\{\hat{I}(Y_a; Y_b)\}$. |
| for $(a,b) \in \hat{T}$ do |
| $\hat{P}^{\text{tree}}(Y_a, Y_b) \leftarrow \hat{P}(Y_a, Y_b)$. |
| end for |

If for each $(a,b) \in \hat{G}_U$, the following quantity

$$\min_{k,l \in \gamma} \|\hat{P}^{\text{spect}}(Y_a|Y_b = k, Y_{N(a)\setminus b} = y, H = h) - \hat{P}^{\text{spect}}(Y_a|Y_b = l, Y_{N(a)\setminus b} = y, H = h)\|_1,$$

is below a certain threshold, for some $y \in \gamma^{|N(a)\setminus b|}$, then it is removed from $\hat{G}_U$, and we obtain $\hat{G}_h$ in this manner. A similar test was used for graphical model selection (i.e., not a mixture model) in (Bresler et al., 2008). We note that we can obtain sample complexity results for the above test, on lines of the analysis in Section 4.1 and this method is efficient when the maximum degree in $G_U$ is small.

## B Extension to Graphs with Sparse Local Separators

### B.1 Graphs with Sparse Local Separators

We now extend the analysis to the setting where the graphical model mixture has the union graph $G_U$ with sparse local separators, which is a weaker criterion than having sparse exact separators. We now provide the definition of a local separator. For detailed discussion, refer to (Anandkumar et al., 2012c).

For $\gamma \in \mathbb{N}$, let $B_\gamma(i; G)$ denote the set of vertices within distance $\gamma$ from $i$ with respect to graph $G$. Let $F_{\gamma,i} := G(B_\gamma(i))$ denote the subgraph of $G$ spanned by $B_\gamma(i; G)$, but in addition, we retain the nodes not in $B_\gamma(i)$ (and remove the corresponding edges).

**Definition 1 ($\gamma$-Local Separator)** Given a graph $G$, a $\gamma$-local separator $S_{\text{local}}(i,j; G, \gamma)$ between $i$ and $j$, for $(i,j) \notin G$, is a minimal vertex separator with respect to the subgraph $F_{\gamma,i}$. In addition, the parameter $\gamma$ is referred to as the path threshold for local separation. A graph is said to be $\eta$-locally separable, if

$$\max_{(i,j) \notin G} |S_{\text{local}}(i,j; G, \gamma)| \leq \eta. \quad (22)$$

A wide family of graphs possess the above property of sparse local separation, i.e., have a small $\eta$. In addition to graphs considered in the previous section, this additionally includes the family of locally tree-like graphs (including sparse random graphs), bounded degree graphs, and augmented graphs, formed by the union of a bounded degree graph and a locally tree-like graph (e.g. small-world graphs). For detailed discussion, refer to (Anandkumar et al., 2012c).

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\[\text{A minimal separator is a separator of smallest cardinality.}\]
B.2 Regime of Correlation Decay

We consider learning mixtures of graphical models Markov on graphs with sparse local separators. We assume that these models are in the regime of correlation decay, which makes learning feasible via our proposed methods. Technically, correlation decay can be defined in multiple ways and we use the notion of strong spatial mixing (Weitz, 2006). A weaker notion is known as weak spatial mixing.

A graphical model is said to satisfy weak spatial mixing when the conditional distribution at each node $v$ is asymptotically independent of the configuration of a growing boundary (with respect to $v$). It is said to satisfy strong spatial mixing, when the total variation distance between two conditional distributions at each node $v$, due to conditioning on different configurations, depends only on the graph distance between node $v$ and the set where the two configurations differ. We formally define it below and incorporate it to provide learning guarantees. See (Weitz, 2006) for details.

Let $P(Y_v|Y_A;G)$ denote the conditional distribution of node $v$ given a set $A \subset V \setminus \{v\}$ under model $P$ with Markov graph $G$. For some subgraph $F \subset G$, let $P(Y_v|Y_A;F)$ denote the conditional distribution on corresponding to a graphical model Markov on subgraph $F$ instead of $G$, i.e., by setting the potentials of edges (and hyperedges) in $G \setminus F$ to zero. For any two sets $A_1, A_2 \subset V$, let $\text{dist}(A_1, A_2) := \min_{u \in A_1, v \in A_2} \text{dist}(u,v)$ denote the minimum graph distance. Let $B_l(v)$ denote the set of nodes within graph distance $l$ from node $v$ and $\partial B_l(v)$ denote the boundary nodes, i.e., exactly at $l$ from node $v$. Let $F_l(v;G) := G(B_l(v))$ denote the induced subgraph on $B_l(v;G)$. For any vectors $a, b$, let $\|a - b\|_1 := \sqrt{\sum_i |a(i) - b(i)|}$ denote the $\ell_1$ distance between them.

Definition 2 (Correlation Decay) A graphical model $P$ Markov on graph $G = (V,E)$ with $p$ nodes is said to exhibit correlation decay with a non-increasing rate function $\zeta(\cdot) > 0$ if for all $l, p \in \mathbb{N}$,

$$\max_{v \in V} \frac{\|P(Y_v|Y_A = y_A;G) - P(Y_v|Y_A = y_A;F_l(i;G))\|_1}{\|y_A\|_1} = \zeta(\text{dist}(A,\partial B_l(i))). \quad (23)$$

Remarks:

1. In (23), if we consider the marginal distribution of node $v$ instead of its conditional distribution over all sets $A$, then we have a weaker criterion, typically referred to as weak spatial mixing. However, in order to provide learning guarantees, we require the notion of strong mixing.

2. For the class of Ising models (binary variables), the regime of correlation decay can be explicitly characterized, in terms of the maximum edge potential of the model. When the maximum edge potential is below a certain threshold, the model is said to be in the regime of correlation decay. The threshold that can be explicitly characterized for certain graph families. See (Anandkumar et al., 2012c) for derivations.

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8We slightly modify the definition of correlation decay compared to the usual notion by considering models on different graphs, where one is an induced subgraph of the neighborhood of the other graph, instead of models with different boundary conditions.
B.3 Rank Test Under Local Separation

We now provide sufficient conditions for the success of $\text{RankTest}(y^n; \xi, p, \eta, r)$ in Algorithm 1. Note that the crucial difference compared to the previous section is that $\eta$ refers to the bound on local separators in contrast to the bound on exact separators. This can lead to significant reduction in computational complexity of running the rank test for many graph families, since the complexity scales as $O(p^{\eta+2}d^3)$ where $p$ is the number of nodes and $d$ is the cardinality of each node variable.

(B1) **Number of Mixture Components:** The number of components $r$ of the mixture model and dimension $d$ of each node variable satisfy

$$d > r. \quad (24)$$

The mixing weights of the latent factor $H$ are assumed to be strictly positive

$$\pi_H(h) := P(H = h) > 0, \quad \forall h \in [r].$$

(B2) **Constraints on Graph Structure:** Recall that $G_{\cup} = \cup_{h=1}^r G_h$ denotes the union of the Markov graphs of the mixture components and we assume that $G_{\cup}$ is $\eta$-locally separable according to Definition 1, i.e., for the chosen path threshold $\gamma \in \mathbb{N}$, we assume that

$$|S_{\text{local}}(u, v; G_{\cup}, \gamma)| \leq \eta = O(1), \quad \forall (u, v) / \in G_{\cup}.$$  

(B3) **Rank Condition:** We assume that the matrix $M_{u,v,\{s,k\}}$ in (4) has rank strictly greater than $r$ when the nodes $u$ and $v$ are neighbors in graph $G_{\cup} = \cup_{h=1}^r G_h$ and the set satisfies $|S| \leq \eta$. Let $\rho_{\text{min}}$ denote

$$\rho_{\text{min}} := \min_{(u,v) \in G_{\cup}, |S| \leq \eta} \max_{k \in Y \setminus \{S\}} \sigma_{r+1}(M_{u,v,\{S,k\}}) > 0. \quad (25)$$

(B4) **Regime of Correlation Decay:** We assume that all the mixture components $\{P(y|H = h; G_h)\}_{h \in [r]}$ are in the regime of correlation decay according to Definition 2 with rate functions $\{\zeta_h(\cdot)\}_{h \in [r]}$. Let

$$\zeta(\gamma) := 2\sqrt{d} \max_{h \in [r]} \zeta_{h}(\gamma). \quad (26)$$

We assume that the minimum singular value $\rho_{\text{min}}$ in (11) and $\zeta(\gamma)$ above satisfy $\rho_{\text{min}} > \zeta(\gamma)$.

(B5) **Choice of threshold $\xi$:** For $\text{RankTest}$ in Algorithm 1, the threshold $\xi$ is chosen as

$$\xi := \frac{\rho_{\text{min}} - \zeta(\gamma)}{2} > 0,$$

where $\zeta(\gamma)$ is given by (26) and $\rho_{\text{min}}$ is given by (11), and $\gamma$ is the path threshold for local separation on graph $G_{\cup}$.

(B6) **Number of Samples:** Given an $\delta > 0$, the number of samples $n$ satisfies

$$n > n_{\text{LRank}}(\delta; p) := \max \left( \frac{1}{t^2} \left( 2 \log p + \log \delta^{-1} + \log 2 \right), \frac{2}{\rho_{\text{min}} - \zeta(\gamma) - t} \right)^2, \quad (27)$$

where $p$ is the number of nodes, for some $t \in (0, \rho_{\text{min}} - \zeta(\gamma))$. 

20
The above assumptions (B1)–(B6) are comparable to assumptions (A1)–(A5) in Section 3.1.1. The conditions on \( r \) and \( d \) in (A1) and (B1) are identical. The conditions (A2) and (B2) are comparable, with the only difference being that (A2) assumes bound on exact separators while (B2) assumes bound on local separators, which is a weaker criterion. Again, the conditions (A3) and (B3) on the rank of matrices for neighboring nodes are identical. The condition (B4) is an additional condition regarding the presence of correlation decay in the mixture components. This assumption is required for approximate conditional independence under conditioning with local separator sets in each mixture component. In addition, we require that \( \zeta(\gamma) < \rho_{\min} \). In other words, the threshold \( \gamma \) on path lengths considered for local separation should be large enough (so that the corresponding value \( \zeta(\gamma) \) is small). (B5) provides a modified threshold to account for distortion due to the use of local separators and (B6) provides the modified sample complexity.

B.3.1 Success of Rank Tests

We now provide the result on the success of recovering the union graph \( G_\cup := \bigcup_{h=1}^r G_h \) for \( \eta \)-locally separable graphs.

**Theorem 4 (Success of Rank Tests)** The \( \text{RankTest}(y^n; \xi, \eta, r) \) outputs the correct graph \( G_\cup := \bigcup_{h=1}^r G_h \), which is the union of the component Markov graphs, under the assumptions (B1)–(B6) with probability at least \( 1 - \delta \).

**Proof:** See Appendix C. \( \square \)

B.4 Results for Spectral Decomposition Under Local Separation

The \( \text{FindMixtureComponents}(y^n, \hat{G}; r) \) procedure in Algorithm 2 can also be implemented for graphs with local separators, but with the modification that we use local separators \( S_{\text{local}}((a, b), c; \hat{G}) \), as opposed to exact separators, between nodes \( a, b \) and \( c \) under consideration. We prove that this method succeeds in estimating the pairwise marginals of the component model under the following set of conditions. We find that there is additional distortion introduced due to the use of local separators in \( \text{FindMixtureComponents} \) as opposed to exact separators.

B.4.1 Assumptions

In addition to the assumptions (B1)–(B6), we impose the following constraints to guarantee the success of estimating the various mixture components.

(B7) **Full Rank Views of the Latent Factor:** For each node pair \( a, b \in V \), and any subset \( S \subset V \setminus \{a, b\} \) with \( |S| \leq 2\eta \) and \( k \in \mathcal{Y}^{|S|} \), the probability matrix \( M_{(a,b)|H,(S;k)} := \{P(Y_{a,b} = i|H = j, Y_S = k)\}_{i,j} \in \mathbb{R}^{d^2 \times r} \) has rank \( r \).

(B8) **Existence of an Isolated Node:** There exists a node \( u_\star \in V \) which is isolated from all other nodes in \( G_\cup = \bigcup_{h=1}^r G_h \), i.e.

\[
Y_{u_\star} \perp \perp Y_{V \setminus u_\star} | H. \tag{28}
\]

(B9) **Spectral Bounds and Random Rotation Matrix:** Refer to various spectral bounds used to obtain \( K(\delta; p, d, r) \) in Appendix D.3 where \( \delta \in (0, 1) \) is fixed. Further assume that the rotation matrix \( Z \in \mathbb{R}^{r \times r} \) in \( \text{FindMixtureComponents} \) is chosen uniformly over the Stiefel manifold \( \{Q \in \mathbb{R}^{r \times r} : Q^\top Q = I\} \).
(B10) **Number of Samples:** For fixed $\delta \in (0,1)$ and $\epsilon > \epsilon_0$, the number of samples satisfies
\[
n > n_{\text{local-spec}}(\delta, \epsilon; p, d, r) := \frac{4K^2(\delta; p, d, r)}{(\epsilon - \epsilon_0)^2},
\]
where
\[
\epsilon_0 := 2K'(\delta; p, d, r)\zeta(\gamma),
\]
and $K'(\delta; p, d, r)$ and $K(\delta; p, d, r)$ are defined in (57) and (58), and $\zeta(\gamma)$ is given by (26).

The assumptions (B7)-(B9) are identical with (A6)-(A8). In (B10), the bound on the number of samples is slightly worse compared to (A9), depending on the correlation decay rate function $\zeta(\gamma)$. Moreover, the perturbation $\epsilon$ now has a lower bound $\epsilon_0$ in (30), due to the use of local separators in contrast to exact vertex separators. As before, below, we impose additional conditions in order to obtain the correct Chow-Liu tree approximation $T_h$ of each mixture component $P(y|H = h)$.

(B11) **Separation of Mutual Information:** Let $T_h$ denote the Chow-Liu tree corresponding to the model $P(y|H = h)$ when exact statistics are input and let
\[
\vartheta := \min_{h \in [r]} \min_{(a,b) \notin T_h} \min_{(u,v) \in \text{Path}(a,b;T_h)} (I(U_u, U_v|H = h) - I(U_a, U_b|H = h)),
\]
where $\text{Path}(a,b;T_h)$ denotes the edges along the path connecting $a$ and $b$ in $T_h$.

(B12) **Constraint on Distortion:** For function $\phi(\cdot)$ defined in (66) in Appendix D.5 and for some $\tau \in (0,0.5\vartheta)$, let $\epsilon_{\text{tree}} := \phi^{-1}\left(\frac{0.5\vartheta - \tau}{3d}\right) > \epsilon_0$, where $\epsilon_0$ is given by (30). The number of samples is now required to satisfy
\[
n > n_{\text{local-spec}}(\delta, \epsilon_{\text{tree}}; p, d, r),
\]
where $n_{\text{local-spec}}$ is given by (29).

Conditions (B11) and (B12) are identical to (A10) and (A11), except that the required bound $\epsilon_{\text{tree}}$ in (B12) is required to be above the lower bound $\epsilon_0$ in (30).

### B.4.2 Guarantees for Learning Mixture Components

We now provide the result on the success of recovering the tree approximation $T_h$ of each mixture component $P(y|H = h)$ under local separation.

**Theorem 5 (Guarantees for FindMixtureComponents)** Under the assumptions (B1)–(B10), the procedure in Algorithm 2 outputs $\hat{P}_{\text{spect}}(Y_a, Y_b|H = h)$, for $a, b \in V \setminus \{u_s\}$, with probability at least $1 - 4\delta$, such that for all $h \in [r]$, there exists a permutation $\tau(h) \in [r]$ with
\[
\|\hat{P}_{\text{spect}}(Y_a, Y_b|H = h) - P(Y_a, Y_b|H = \tau(h))\|_2 \leq \epsilon.
\]
Moreover, under additional assumptions (B11)-(B12), the method outputs the correct Chow-Liu tree $T_h$ of each component $P(y|H = h)$ with probability at least $1 - 4\delta$.

**Remark:** The sample and computational complexities are significantly improved, since it only depends on the size of local separators (while previously it depended on the size of exact separators).

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9 Assume that the Chow-Liu tree $T_h$ is unique for each component $h \in [r]$ under exact statistics, and this holds for generic parameters.
C. Analysis of Rank Test: Proof of Theorem 1 and 4

Bounds on Empirical Probability: We first recap the result from [Hsu et al., 2009, Proposition 19], which is an application of the McDiarmid’s inequality. Let $\|\cdot\|_2$ the $\ell_2$ norm of a vector.

Proposition 1 (Bound for Empirical Probability Estimates) Given empirical estimates $\hat{P}^n$ of a probability vector $P$ using $n$ i.i.d. samples, we have

$$\mathbb{P}[\|\hat{P}^n - P\|_2 > \epsilon] \leq \exp \left[ -n \left( \epsilon - 1/\sqrt{n} \right)^2 \right], \quad \forall \epsilon > 1/\sqrt{n}. \quad (34)$$

Remark: The bound is independent of the cardinality of the sample space.

This implies concentration bounds for $\hat{M}_{u,v;\{S;k\}}$. Let $\|\cdot\|_2$ and $\|\cdot\|_F$ denote the spectral norm and the Frobenius norms respectively.

Lemma 3 (Bounds for $\hat{M}_{u,v;\{S;k\}}$) Given $n$ i.i.d. samples $y^n$, the empirical estimate $\hat{M}^n_{u,v;\{S;k\}} := [\hat{P}^n|Y_u = i, Y_v = j, Y_S = k]|_{i,j}$ satisfies

$$\mathbb{P}\left[ \max_{k \in [d]} \max_{l \in [S]} |\sigma_l(\hat{M}^n_{u,v;\{S;k\}}) - \sigma_l(M_{u,v;\{S;k\}})| > \epsilon \right] \leq \exp \left[ -n \left( \epsilon - 1/\sqrt{n} \right)^2 \right], \quad \forall \epsilon > 1/\sqrt{n}. \quad (35)$$

Proof: Using proposition 1, we have

$$\mathbb{P}\left[ \max_{k \in [S]} \|\hat{P}^n(Y_u, Y_v, Y_S = k) - P(Y_u, Y_v, Y_S = k)\|_2 > \epsilon \right] \leq \exp \left[ -n \left( \epsilon - 1/\sqrt{n} \right)^2 \right], \quad \epsilon > 1/\sqrt{n}. \quad (36)$$

In other words,

$$\mathbb{P}\left[ \max_{k \in [S]} \|\hat{M}^n_{u,v;\{S;k\}} - M_{u,v;\{S;k\}}\|_F > \epsilon \right] \leq \exp \left[ -n \left( \epsilon - 1/\sqrt{n} \right)^2 \right], \quad \epsilon > 1/\sqrt{n}. \quad (37)$$

Since $\|A\|_2 \leq \|A\|_F$ for any matrix $A$ and applying the Weyl’s theorem, we have the result. From Lemma 1 and Lemma 3 it is easy to see that

$$\mathbb{P}[\hat{G}^n_{\cup} \neq G_{\cup}] \leq 2p^2 \exp \left[ -n \left( \rho_{\min}/2 - 1/\sqrt{n} \right)^2 \right],$$

and we have the result. Similarly, we have Theorem 4 from Lemma 1 and Lemma 3. \hfill \Box

D. Analysis of Spectral Decomposition: Proof of Theorem 2

D.1 Analysis Under Exact Statistics

We now prove the success of FindMixtureComponents under exact statistics. We first consider three sets $A_1, A_2, A_3 \subseteq V$ such that $N[A_i;G_{\cup}] \cap N[A_j;G_{\cup}] = \emptyset$ for $i,j \in [3]$ and $G_{\cup} := \cup_{h \in [r]} G_h$ is the union of the Markov graphs. Let $S \subseteq V \setminus \cup_i A_i$ be a multiway separator set for $A_1, A_2, A_3$ in graph $G_{\cup}$. For $A_i$, $i \in \{1,2,3\}$, let $U_i \in \mathbb{R}^{d_i \times r}$ be a matrix such that $U_i^T M_{A_i;H_i;\{S;k\}}$ is invertible, for a
fixed \( k \in \mathcal{Y} \). Then \( U_1^T M_{A_1,A_2,(S;k)} U_2 \) is invertible, and for all \( \mathbf{m} \in \mathbb{R}^{d_{A_3}} \), the observable operator \( \widetilde{C} (\mathbf{m}) \in \mathbb{R}^{r \times r} \), given by

\[
\widetilde{C} (\mathbf{m}) := \left( U_1^T \left( \sum_{q} m(q) M_{A_1,A_2,(S;k),(A_3;q)} \right) U_2 \right) \left( U_1^T M_{A_1,A_2,(S;k)} U_2 \right)^{-1}. \tag{38}
\]

Note that the above operator is computed in \textbf{SpecDecom} procedure. We now provide a generalization of the result in [Anandkumar et al. 2012b].

**Lemma 4 (Observable Operator)** Under assumption (A6), the observable operator in (38) satisfies

\[
\widetilde{C} (\mathbf{m}) = (U_1^T M_{A_1|H,(S;k)}) \text{Diag} \left( M_{A_3|H,(S;k)}^\top \mathbf{m} \right) \left( U_1^T M_{A_1|H,(S;k)} \right)^{-1}. \tag{39}
\]

In particular, the \( r \) roots of the polynomial \( \lambda \mapsto \det(\widetilde{C} (\mathbf{m}) - \lambda I) \) are \( \{ \langle \mathbf{m}, M_{A_3|H,(S;k)} \mathbf{e}_j \rangle : j \in [r] \} \).

**Proof:** We have

\[
U_1^T M_{A_1,A_2,(S;k)} U_2 = (U_1^T M_{A_1|H,(S;k)}) \text{Diag} (\pi_{H,(S;k)}) (M_{A_2|H,(S;k)}^\top U_2)
\]
on lines of (39), which is invertible by the assumptions on \( U_1, U_2 \) and Assumption (A6). Similarly,

\[
U_1^T M_{A_1,A_2,(S;k),(A_3;q)} U_2 = (U_1^T M_{A_1|H,(S;k)}) \text{Diag} (\pi_{H,(S;k),(A_3;q)}) (M_{A_2|H,(S;k)}^\top U_2),
\]
and we have the result. \( \square \)

The above result implies that we can recover the matrix \( M_{A|H,(S;k)} \) for any set \( A \subset V \), by using a suitable reference node, a witness and a separator set. We set the isolated node \( u_* \) as the reference node (set \( A_1 \) in the above result). Since we focus on recovering the edge marginals of the mixture components, we consider each node pair \( a, b \in V \setminus \{ u_* \} \) (set \( A_3 \) in the above result), and any node \( c \notin \mathcal{N}(a; G_U) \cup \mathcal{N}(b; G_U) \) (set \( A_2 \) in the above result), where \( G_U := \cup_{h \in [r]} G_h \), as described in \textbf{FindMixtureComponents}. Thus, we are able to recover \( M_{a,b|H,(S;k)} \) under exact statistics. Since \( Y_S \) are observed, we have the knowledge of \( P(Y_S = k) \), and can thus recover \( M_{a,b|H} \) as desired. The spectral decompositions of different groups are aligned since we use the same node \( u_* \), and since \( u_* \) is isolated in \( G_U \), fixing the variables \( Y_S = k \) has no effect on the conditional distribution of \( Y_{u_*} \), i.e., \( P(Y_{u_*}|H,Y_S = k) = P(Y_{u_*}|H) \). Since we recover the edge marginals \( M_{a,b|H} \) correctly we can recover the correct tree approximation \( T_h \), for \( h \in [r] \).

**D.2 Analysis of SpecDecom(\( u, v, w; S \))**

We first consider the success of Procedure \textbf{SpecDecom}(\( u, v, w; S \)) for estimating the statistics of \( w \) using node \( u \in V \) as the reference node (which is conditionally independent of all other nodes given \( H \)) and witness \( v \in V \) and separator set \( S \). We will use this to provide sample complexity results on \textbf{FindMixtureComponents} using union bounds. The proof borrows heavily from [Anandkumar et al., 2012b].

Recall that \( \widehat{U}_u \) is the set of top \( r \) left orthonormal singular vectors of \( \widehat{M}^n_{u,v,(S;k)} \) and \( \widehat{V}_v \) as the right orthonormal vectors. For \( l \in [r] \), let \( \mathbf{m}_l = \widehat{U}_u \mathbf{z}_l \), where \( \mathbf{z}_l \) is uniformly distributed in \( S^{r-1} \) and...
\( \hat{U}_w \) is the top \( r \) left singular vectors of \( \hat{M}^n_{w,u,(S;k)} \). By Lemma \[13\] we have that \( U_u^\top M_{u,v,(S;k)} V_v \) is invertible. Recall the definition of the observable operator in \[38\]

\[
\tilde{C}_t := \tilde{C}(m_t) = \hat{U}_u^\top \left( \sum_q m_t(q) M_{u,v,(S;k),w;q} \right) \hat{V}_v \left( U_u^\top M_{u,v,(S;k)} V_v \right)^{-1},
\]

where exact matrices \( M \) are used. Denote \( \tilde{C}_t \) when the sample versions \( \hat{M}^n \) are used

\[
\hat{C}_t := \hat{U}_u^\top \left( \sum_q m_t(q) \hat{M}_{u,v,(S;k),w;q}^n \right) \hat{V}_v \left( U_u^\top \hat{M}_{u,v,(S;k)}^n V_v \right)^{-1},
\]

We have the following result.

**Lemma 5 (Bounds for \( \|\hat{C}_t - \tilde{C}_t\|_2 \))** The matrices \( \hat{C}_t \) and \( \tilde{C}_t \) defined in \[40\] and \[41\] satisfy

\[
\|\hat{C}_t - \tilde{C}_t\|_2 \leq \frac{2 \| \sum_q m_t(q) (\hat{M}_{u,v,(S;k),w;q}^n - M_{u,v,(S;k),w;q}) \|_2}{\sigma_r(M_{u,v,(S;k)})} + \frac{2 \| \sum_q m_t(q) M_{u,v,(S;k),w;q} \|_2 \| \hat{M}_{u,v,(S;k)}^n - M_{u,v,(S;k)} \|_2}{\sigma_r(M_{u,v,(S;k)})^2}.
\]

**Proof:** Using Lemma \[14\] and Lemma \[1\].

We now provide perturbation bounds between estimated matrix \( \hat{M}_{w|H,(S;k)} \) and the true matrix \( M_{w|H,(S;k)} \). Define

\[
\beta(w) := \min_{k \in [b]} \min_{i \in [r]} \min_{j \neq j'} \| \langle \mathbf{z}^{(i)}, \hat{U}_w M_{w|H,(S;k)} (\mathbf{e}_j - \mathbf{e}_{j'}) \rangle \|
\]

\[
\lambda_{\text{max}}(w) := \max_{i,j \in [r]} | \langle \mathbf{z}^{(i)}, \hat{U}_w M_{w|H,(S;k)} \mathbf{e}_j \rangle |,
\]

where \( \mathbf{z}_t \) is uniformly distributed in \( S^{r-1} \).

**Lemma 6 (Relating \( \hat{M}_{w|H,(S;k)} \) and \( M_{w|H,(S;k)} \))** The estimated matrix \( \hat{M}_{w|H,(S;k)} \) using samples and the true matrix \( M_{w|H,(S;k)} \) satisfy, for all \( j \in [r] \),

\[
\| \hat{M}_{w|H,(S;k)} \mathbf{e}_j - M_{w|H,(S;k)} \mathbf{e}_j \|_2 \leq 2 \| M_{w|H,(S;k)} \mathbf{e}_j \|_2 \cdot \frac{\| \hat{M}_{w,u,(S;k)}^n - M_{u,u,(S;k)} \|_2}{\sigma_r(M_{u,u,(S;k)})} + \left( 12 \sqrt{\tau} \cdot \kappa(M_{u,H})^2 + 256r^2 \cdot \kappa(M_{u,H})^4 \cdot \lambda_{\text{max}}(w) / \beta(w) \right)^{1/2} \cdot \| \hat{C}_t - \tilde{C}_t \|_2.
\]

**Proof:** Define a matrix \( R := \hat{U}_u^\top M_{u,H} \text{Diag} (\| \hat{U}_u^\top M_{u,H} \mathbf{e}_1 \|_2, \ldots, \| \hat{U}_u^\top M_{u,H} \mathbf{e}_r \|_2)^{-1} \). Note that \( R \) has unit norm columns and \( R \) diagonalizes \( \tilde{C}_t \), i.e.,

\[
R^{-1} \tilde{C}_t R = \text{Diag}(M_{w|H,(S;k)} \mathbf{z}_t).
\]

Using the fact that for any stochastic matrix \( d \times r \) matrix \( A \), \( \| A \|_2 \leq \sqrt{\tau} \| A \|_1 = \sqrt{\tau} \) and Lemma \[17\] we have

\[
\| R^{-1} \|_2 \leq 2 \kappa(\hat{U}_u^\top M_{u,H}), \quad \kappa(R) \leq 4 \kappa(M_{u,H}).
\]
From above and by Lemma 14, there exist a permutation $\tau$ on $[r]$ such that, for all $j, l \in [r]$, 
\[
|\hat{\lambda}^{(l)}(j) - \lambda^{(l)}(\tau(j))| \leq \left(3\kappa(R) + 16r^{1.5} \cdot \kappa(R) \cdot R^{-1} \|\beta(w)\|/\beta(w)\right) \cdot \|\hat{C}_l - \tilde{C}_l\|_2
\]
\[
\leq \left(12\kappa(M_{u|H})^2 + 256r^{1.5} \cdot \kappa(M_{u|H})^4 \cdot \lambda_{\max}(w)/\beta(w)\right) \cdot \|\hat{C}_l - \tilde{C}_l\|_2, \tag{46}
\]
where $\beta(w)$ and $\lambda_{\max}(w)$ are given by (43) and (44). Let $\hat{\nu}^{(j)} := (\hat{\lambda}^{(1)}(j), \hat{\lambda}^{(2)}(j), \ldots, \hat{\lambda}^{(r)}(j)) \in \mathbb{R}^r$ be the row vector corresponding to the $j$th row of $\hat{\Lambda}$ and $\tilde{\nu}^{(j)} := (\lambda^{(1)}(j), \lambda^{(2)}(j), \ldots, \lambda^{(r)}(j)) \in \mathbb{R}^r$. Observe that $\hat{\nu}^{(j)} = Z \tilde{U}_{w|H,\{S:k\}}^T M_{w|H,\{S:k\}} \tilde{e}_j$. By the orthogonality of $Z$, the fact $\|\tilde{v}\|_2 \leq \sqrt{r} \|\tilde{v}\|_\infty$ for $\tilde{v} \in \mathbb{R}^r$, and the above inequality,
\[
\|Z^{-1}\hat{\nu}^{(j)} - \tilde{U}_{w|H,\{S:k\}}^T M_{w|H,\{S:k\}} \tilde{e}_j\|_2
\]
\[
= \|Z^{-1}(\hat{\nu}^{(j)} - \tilde{\nu}^{(\tau(j))})\|_2
\]
\[
= \|\hat{\nu}^{(j)} - \tilde{\nu}^{(\tau(j))}\|_2
\]
\[
\leq \sqrt{r} \cdot \|\hat{\nu}^{(j)} - \tilde{\nu}^{(\tau(j))}\|_\infty
\]
\[
\leq \left(12\sqrt{r} \cdot \kappa(M_{u|H})^2 + 256r^{1.5} \cdot \kappa(M_{u|H})^4 \cdot \lambda_{\max}(w)/\beta(w)\right) \cdot \|\hat{C}_l - \tilde{C}_l\|_2.
\]

By Lemma 13 (as applied to $\hat{M}_{u,w,\{S:k\}}^n$ and $M_{u,w,\{S:k\}}$), we have
\[
\|\hat{M}_{w|H,\{S:k\}} e_j - M_{w|H,\{S:k\}} e_{\tau(j)}\|_2 \leq \|Z^{-1}\hat{\nu}^{(j)} - \tilde{U}_{w|H,\{S:k\}}^T M_{w|H,\{S:k\}} \tilde{e}_j\|_2
\]
\[
+ 2\|M_{w|H,\{S:k\}} e_{\tau(j)}\|_2 \cdot \frac{\|\hat{M}_{u,w,\{S:k\}}^n - M_{u,w,\{S:k\}}\|_2}{\sigma_\tau(M_{u,w,\{S:k\}})}. \tag{47}
\]

D.3 Analysis of FindMixtureComponents

We now provide results for Procedure FindMixtureComponents by using the previous result, where $w$ is set to each node pair $a, b \in V \setminus \{u_*\}$. We condition on the event that $G_\cup = G_\cup$, where $G_\cup := \cup_{h \in [r]} G_h$ is the union of the component graph.

We now give concentration bounds for $\beta$ and $\lambda_{\max}$ in (43) and (44). Define
\[
\alpha_{\min} := \min_{a,b \in V \setminus \{u_*\}} \min_{k \in \mathbb{N}, |S| \leq 2\eta} \min_{i \neq i'} \|M_{(a,b)|H,\{S:k\}} (e_i - e_{i'})\|_2, \tag{48}
\]
\[
\alpha_{\max} := \max_{a,b \in V \setminus \{u_*\}} \max_{k \in \mathbb{N}, |S| \leq 2\eta} \max_{j \in [r]} \|M_{(a,b)|H,\{S:k\}} e_j\|_2, \tag{49}
\]
and let
\[
\alpha := \frac{\alpha_{\max}}{\alpha_{\min}}. \tag{50}
\]

Lemma 7 (Bounds for $\beta$ and $\lambda_{\max}$) Fix $\delta \in (0, 1)$, given any $a, b \in V \setminus \{u_*\}$ and any set $S \subset V \setminus \{a, b, u_*\}$ with $|S| \leq 2\eta$, we have with probability at least $1 - \delta$,
\[
\beta(a, b) \geq \frac{\alpha_{\min} \cdot \delta}{2 \sqrt{er(t)} \cdot r p^2(pd)^2 \eta}
\]
\[
\lambda_{\max}(a, b) \leq \frac{\alpha_{\max}}{\sqrt{T}} \left(1 + \sqrt{2\ln(r^2p^2(pd)^2\eta/\delta)}\right) \tag{52}
\]
This implies that with probability at least $1 - 2\delta$,

$$\frac{\lambda_{\max}(a, b)}{\beta(a, b)} \geq \frac{\sqrt{c\alpha}}{\delta} r^3 p^2(pd)^{2n} \left(1 + \sqrt{2\ln(r^2p^2(pd)^{2n}/\delta)}\right),$$  \hspace{1cm} (53)

where $\alpha$ is given by (50).

Similarly, we have bounds on $\|\tilde{M}_{u,a,b,\{S;k\}} - M_{u,a,b,\{S;k\}}\|_2$ using Lemma 8 and union bound.

**Proposition 2** ($\|\tilde{M}_{u,a,b,\{S;k\}} - M_{u,a,b,\{S;k\}}\|_2$) With probability at least $1 - \delta$, we have, for all $a, b \in V \setminus \{u_*\}$, $S \subset V \setminus \{a, b, u_*\}$, $|S| \leq 2\eta$,

$$\|\tilde{M}_{u,a,b,\{S;k\}} - M_{u,a,b,\{S;k\}}\|_2 \leq \frac{1}{\sqrt{n}} \left(1 + \sqrt{\log \left(\frac{p^{2\eta+2}d^{2\eta}}{\delta}\right)}\right).$$  \hspace{1cm} (54)

Define $\rho_{1,\text{min}}', \rho_{2,\text{min}}'$ and $\rho_{\text{max}}'$ as

$$\rho_{1,\text{min}}' := \min_{S \subset V \setminus \{u_*\}} \min_{v \in V \setminus \{u_*\}} \min_{|S| \leq 2\eta, k \in \{\cdot\}} \sigma_r \left(M_{u,v,\{S;k\}}\right),$$  \hspace{1cm} (55)

$$\rho_{2,\text{min}}' := \min_{S \subset V \setminus \{u_*\}} \min_{a,b \in V \setminus \{u_*\}} \min_{|S| \leq 2\eta, k \in \{\cdot\}} \sigma_r \left(M_{u,a,b,\{S;k\}}\right).$$  \hspace{1cm} (56)

Using the above defined constants, define

$$K' (\delta; p, d, r) := 1024 \cdot \kappa(M_{u|H})^4 \cdot \frac{\sqrt{c\alpha}}{\delta \rho_{1,\text{min}}'} r^5 p^2(pd)^{2n} \left(1 + \sqrt{2\ln(r^2p^2(pd)^{2n}/\delta)}\right)$$

$$+ 48 \frac{\sqrt{r}}{\rho_{1,\text{min}}'} \cdot \kappa(M_{u|H})^2 + \frac{2\alpha_{\text{max}}}{\rho_{2,\text{min}}'},$$  \hspace{1cm} (57)

and

$$K (\delta; p, d, r) := K' (\delta; p, d, r) \left(1 + \sqrt{\log \left(\frac{p^{2\eta+2}d^{2\eta}}{\delta}\right)}\right).$$  \hspace{1cm} (58)

We can now provide the final bound on distortion of estimated statistics using all the previous results.

**Lemma 8** (Bounds for $\|\tilde{M}_{a,b|H,\{S;k\}} e_j - M_{a,b|H,\{S;k\}} e_{\tau(j)}\|_2$) For any $a, b \in V \setminus \{u_*\}$, $k \in \mathcal{Y}^{|S|}$, $j \in [r]$, there exists a permutation $\tau(j) \in [r]$ such that, conditioned on event that $\tilde{G}_U = G_U$, with probability at least $1 - 3\delta$,

$$\|\tilde{M}_{a,b|H,\{S;k\}} e_j - M_{a,b|H,\{S;k\}} e_{\tau(j)}\|_2 \leq \frac{K (\delta; p, d, r)}{\sqrt{n}}.$$  \hspace{1cm} (59)

This implies

$$\|\tilde{M}_{a,b|H} e_j - M_{a,b|H} e_{\tau(j)}\|_2 \leq \frac{K (\delta; p, d, r)}{\sqrt{n}} + \frac{K (\delta; p, d, r)}{K' (\delta; p, d, r)\sqrt{n}} \leq \frac{2K (\delta; p, d, r)}{\sqrt{n}}.$$  \hspace{1cm} (60)
Results on Random Rotation Matrix: We also require the following result from Anandkumar et al., 2012b. The standard inner product between vectors \( \vec{u} \) and \( \vec{v} \) is denoted by \( \langle \vec{u}, \vec{v} \rangle = \vec{u}^\top \vec{v} \). Let \( \sigma_i(A) \) denote the \( i^{th} \) largest singular value of a matrix \( A \). Let \( \mathbb{S}^{m-1} := \{ \vec{u} \in \mathbb{R}^m : \|\vec{u}\|_2 = 1 \} \) denote the unit sphere in \( \mathbb{R}^m \). Let \( \vec{e}_i \in \mathbb{R}^d \) denote the \( i^{th} \) coordinate vector where the \( i^{th} \) entry is 1, and the rest are zero.

**Lemma 9** Fix any \( \delta \in (0,1) \) and matrix \( A \in \mathbb{R}^{m \times n} \) (with \( m \leq n \)). Let \( \vec{\theta} \in \mathbb{R}^m \) be a random vector distributed uniformly over \( \mathbb{S}^{m-1} \).

1. \( \Pr \left[ \min_{i \neq j} |\langle \vec{\theta}, A(\vec{e}_i - \vec{e}_j) \rangle| > \frac{\sqrt{2} \sigma_m(A) \cdot \delta}{\sqrt{mn} (\binom{m}{2})} \right] \geq 1 - \delta. \)

2. \( \Pr \left[ \forall i \in [m], |\langle \vec{\theta}, A\vec{e}_i \rangle| \leq \frac{\|A\vec{e}_i\|_2}{\sqrt{m}} \left( 1 + \sqrt{2 \ln(m/\delta)} \right) \right] \geq 1 - \delta. \)

D.4 Improved Results for Tree Mixtures

We now consider a simplified version of \texttt{FindMixtureComponents} by limiting to estimation of pairwise marginals only on the edges of \( \hat{G}_{\cup} \), where \( \hat{G}_{\cup} \) is the estimate of \( G_{\cup} := \cup_{h \in [\nu]} G_h \), which is the union of the component graph, as well as constructing the Chow-Liu trees \( \hat{T}_h \) as subgraphs of \( \hat{G}_{\cup} \). Thus, instead of considering each node pair \( a, b \in V \setminus \{ u_* \} \), we only need to choose \( (a, b) \in \hat{G}_{\cup} \). Moreover, instead of considering \( S \subset V \setminus \{ a, b, u_* \} \), we can follow the convention of choosing \( S \subset N(a; \hat{G}_{\cup}) \cup N(b; \hat{G}_{\cup}) \), and this changes the definition of \( \alpha_{\min}, \alpha_{\max}, \rho'_{1, \min}, \rho'_{2, \min} \) and so on. For all \( (a, b) \in G_{\cup} \), let

\[
\Delta_2 := \max_{(a, b) \in G_{\cup}} |N(a; G_{\cup}) \cup N(b; G_{\cup})|. \tag{61}
\]

We have improved bounds for \( \beta \) and \( \lambda_{\max} \) defined in (43) and (44), when \( \Delta_2 \) is small.

**Lemma 10 (Improved Bounds for \( \beta \) and \( \lambda_{\max} \))** Fix \( \delta \in (0,1) \), when \( |S| \leq 2\eta \) and \( S \subset N(a; G_{\cup}) \cup N(b; G_{\cup}) \), with probability at least \( 1 - \delta \),

\[
\beta(w) \geq \frac{\sqrt{2} \alpha_{\min} \delta}{\sqrt{\binom{r+1}{2} r p^2 d^2 n \Delta_2^{2\eta}}} \tag{62}
\]

\[
\lambda_{\max}(w) \leq \frac{\alpha_{\max}}{\sqrt{r}} \left( 1 + \sqrt{2 \ln(r^2 p^2 d^2 n \Delta_2^{2\eta} / \delta)} \right) \tag{63}
\]

We can substitute the above result to obtain a better bound \( R^{\text{tree}}(\delta; p, d, r) \) for learning tree mixtures.

D.5 Analysis of Tree Approximations: Proof of Theorem 3

We now relate the perturbation of probability vector to perturbation of the corresponding mutual information (Cover and Thomas, 2006). Recall that for discrete random variables \( X, Y \), the mutual information \( I(X; Y) \) is related to their entropies \( H(X, Y), H(X) \) and \( H(Y) \) as

\[
I(X; Y) = H(X) + H(Y) - H(X, Y), \tag{64}
\]
and the entropy is defined as
\[ H(X) := - \sum_{x \in \mathcal{X}} P(X = x) \log P(X = x), \] (65)
where \( \mathcal{X} \) is the sample space of \( X \). We recall the following result from Shamir et al. (2008). Define function \( \phi(x) \) for \( x \in \mathbb{R}^+ \) as
\[
\phi(x) = \begin{cases} 
0, & x = 0, \\
-x \log x, & x \in (0, 1/e), \\
1/e, & \text{o.w.}
\end{cases}
\] (66a)
(66b)
(66c)

**Proposition 3** For any \( a, b \in [0, 1] \),
\[ |a \log a - b \log b| \leq \phi(|a - b|), \] (67)
for \( \phi(\cdot) \) defined in (66).

We can thus prove bounds on the estimated mutual information \( \hat{I}_{\text{spect}}(\cdot) \) using statistics \( \hat{P}_{\text{spect}}(\cdot) \) obtained from spectral decomposition.

**Proposition 4 (Bounding \( |\hat{I}_{\text{spect}}(\cdot) - I(\cdot)| \))** Under the event that \( \| \hat{P}_{\text{spect}}(Y_a, Y_a|H = h) - P(Y_a, Y_a|H = h) \|_2 \leq \epsilon \), we have that
\[ |\hat{I}_{\text{spect}}(Y_a; Y_a|H = h) - I(Y_a; Y_a|H = h)| \leq 3d\phi(\epsilon). \] (68)

For success of Chow-Liu algorithm, it is easy to see that the algorithm finds the correct tree when the estimated mutual information quantities are within half the minimum separation \( \vartheta \) defined in (20). This is because the only wrong edges in the estimated tree \( \hat{T}_h \) are those that replace a certain edge in the original tree \( T_h \), without violating the tree constraint. Similar ideas have been used by Tan et al. (2011) for deriving error exponent bounds for the Chow-Liu algorithm. Define
\[ \epsilon_{\text{tree}} := \phi^{-1} \left( \frac{0.5\vartheta - \tau}{3d} \right). \] (69)

Thus, using the above result and assumption (A11) implies that we can estimate the mutual information to required accuracy to obtain the correct tree approximations.

**E Analysis Under Local Separation Criterion**

**E.1 Rank Tests Under Approximate Separation**

We now extend the results of the previous section when approximate separators are employed in contrast to exact vertex separators. Let \( S := S_{\text{local}}(u, v; G, \gamma) \) denote a local vertex separator between any non-neighboring nodes \( u \) and \( v \) in graph \( G \) under threshold \( \gamma \). We note the following result on the probability matrix \( M_{u,v,\{S,k\}} \) defined in (4).
Lemma 11 (Rank Upon Approximate Separation) Given a $r$-mixture of graphical models with $G = \bigcup_{h=1}^{r} G_h$, for any nodes $u,v \in V$ such that $\mathcal{N}[u] \cap \mathcal{N}[v] = \emptyset$ and $S := S_{\text{local}}(u,v;G,\gamma)$ be any separator of $u$ and $v$ on $G$, the probability matrix $M_{u,v;\{S;k\}} := \{P[Y_u = i, Y_v = j, Y_S = k]\}_{i,j}$ has effective rank at most $r$ for any $k \in \mathcal{Y}^{|S|}$

$$\text{Rank} \left( M_{u,v;\{S;k\}}; \zeta(\gamma) \right) \leq r, \quad \forall k \in \mathcal{Y}^{|S|}, (u,v) \notin G,$$ (70)

where $\zeta(\gamma) := 2\sqrt{d} \max_{h \in [r]} \zeta_h(\gamma)$, and $\zeta_h(\cdot)$ is the correlation decay rate function in (23) corresponding to the model $P(y|H = h)$ and $\gamma$ is the path threshold for local vertex separators.

Notation: For convenience, for any node $v \in V$, let $P(Y_v|H = h) := P(Y_v|H = h; G_h)$ denote the original component model Markov on graph $G_h$, and let $P(Y_v)$ denote the corresponding marginal distribution of $Y_v$ in the mixture. Let $\hat{P}^\gamma(\gamma|H = h) := P(Y_v|H = h; F_{\gamma,h})$ denote the component model Markov on the induced subgraph $F_{\gamma,h} := G_h(B_\gamma(v))$, where $B_\gamma(v;G_h)$ is the $\gamma$-neighborhood of node $v$ in $G_h$. In other words, we limit the model parameters up to $\gamma$ neighborhood and remove rest of the edges to obtain $\hat{P}^\gamma(Y_v|H = h)$.

Proof: We first claim that

$$\|M_{u|v;\{S;k\}} - M_{u|H_{v};\{S;k\}}M_{H|v;\{S;k\}}\|_2 \leq \zeta(\gamma).$$ (71)

Note the relationship between the joint and the conditional probability matrices:

$$M_{u,v;\{S;k\}} = M_{u|v;\{S;k\}} \text{ Diag}(\pi_v;\{S;k\}),$$ (72)

where $\pi_v;\{S;k\} := [P(Y_v = i, Y_S = k)]_i^T$ is the probability vector and Diag(·) is the diagonal matrix with the corresponding probability vector as the diagonal elements. Assuming (71) holds and applying (72), we have that

$$\|M_{u,v;\{S;k\}} - M_{u|H_{v};\{S;k\}}M_{H|v;\{S;k\}} \text{ Diag}(\pi_v;\{S;k\})\|_2 \leq \|\text{Diag}(\pi_v;\{S;k\})\|_2 \zeta(\gamma) \leq \zeta(\gamma),$$ (73)

since $\|\text{Diag}(\pi_v;\{S;k\};G)\|_2 \leq \|\text{Diag}(\pi_v;\{S;k\};G)\|_F = \|\pi_v;\{S;k\};G\|_2 \leq 1$ for a probability vector. From Weyl’s theorem, assuming that (73) holds, we have

$$\text{Rank} \left( M_{u,v;\{S;k\}}; \zeta(\gamma) \right) \leq \min(r,d) = r,$$

since we assume $r < d$ (assumption (B1) in Section B.3). Note that $\text{Rank}(A;\xi)$ denotes the effective rank, i.e., the number of singular values of $A$ which are greater than $\xi \geq 0$.

We now prove the claim in (71). Since $G = \bigcup_{h=1}^{r} G_h$, we have that the resulting set $S := S_{\text{local}}(u,v;G,\gamma)$ is also a local separator on each of the component subgraphs $\{G_h\}_{h \in [r]}$ of $G$, for all sets $A,B \subset V$ such that $\mathcal{N}[u;G] \cap \mathcal{N}[v;G] = \emptyset$. Thus, we have that for all $k \in \mathcal{Y}^{|S|}$, $y_v \in \mathcal{Y}$, $h \in [r]$,

$$\hat{P}^\gamma(Y_u|Y_v = y_v, Y_S = k, H = h) = \hat{P}^\gamma(Y_u|Y_S = k, H = h).$$ (74)

The statement in (74) is due to the fact that the nodes $u$ and $v$ are exactly separated by set $S$ in the subgraph $F_{\gamma,h}(u)$.
By assumption (B4) on correlation decay we have that
\[ \|P(Y_u | Y_v = y_v, Y_S = k, H = h) - P(Y_u | Y_v = y_v, Y_S = k, H = h)\|_1 \leq \zeta_h(\gamma), \]
for all \( y_v \in \mathcal{Y}, k \in \mathcal{Y}^{|S|} \) and \( h \in [r] \). Similarly, we also have
\[ \|P(Y_u | Y_S = k, H = h) - P(Y_u | Y_S = k, H = h)\|_1 \leq \zeta_h(\gamma), \]
which implies that
\[ \|P(Y_u | Y_v = y_v, Y_S = k, H = h) - P(Y_u | Y_v = y_v, Y_S = k, H = h)\|_1 \leq 2\zeta_h(\gamma), \]
for all \( y_v \in \mathcal{Y}, k \in \mathcal{Y}^{|S|} \) and \( h \in [r] \), and thus,
\[ \|M_{u,v,\{S;k\}} - M_{u,H,\{S;k\}} \|_1 \leq 2 \max_{h \in [r]} \zeta_h(\gamma), \]
where \( \|A\|_1 \) of a matrix is the maximum column-wise absolute sum. Since \( \|A\|_1 \leq \sqrt{d}\|A\|_1 \), (71) follows.

**E.2 Spectral Decomposition Under Local Separation**

We now extend the above analysis of spectral decomposition when a local separator is used instead of approximate separators. For simplicity consider nodes \( u_*, a, b, c \in V \) (the same results can also be proven for larger sets), where \( u_* \) is an isolated node in \( G_\cup \), \( a, b \in V \setminus \{u_*\} \), \( c \notin N[a; G_\cup] \cup N[b; G_\cup] \) and let \( S := S_{\text{local}}((a, b), c; G_\cup) \) be a local separator in \( G_\cup \) separating \( a, b \) from \( c \). Since we have
\[ Y_{u_*} \perp \perp Y_{V \setminus \{u_*\}|H}, \]
the following decomposition holds
\[ M_{u_*,c,\{S;k\}} = M_{u_*,H} \text{Diag}(\pi_{H,\{S;k\}}) M_{c,H,\{S;k\}}^\top. \]

However, the matrix \( M_{u_*,c,\{S;k\}),(a,b);q} \) no longer has a similar decomposition. Instead define
\[ \bar{M}_{u_*,c,\{S;k\}),(a,b);q} := M_{u_*,\{H\} \text{Diag}(\pi_{H,\{S;k\}),(a,b);q} \). \]

Define the observable operator, on lines of (68), based on \( \bar{M} \) above rather than the actual probability matrix \( M \), as
\[ \bar{C}(m) := \left( U_1^\top \left( \sum_q m(q) \bar{M}_{u_*,c,\{S;k\},(a,b);q} \right) U_2 \right) \left( U_1^\top M_{u_*,c,\{S;k\}} U_2 \right)^{-1}, \]
where \( U_1 \) is a matrix such that \( U_1^\top M_{u_*,H} \) is invertible and \( U_2 \) is such that \( U_2^\top M_{c,H,\{S;k\}} \) is invertible. On lines of Lemma 4 we have that
\[ \bar{C}(m) = \left( U_1^\top M_{u_*,H} \right) \text{Diag} \left( M_{(a,b);H,\{S;k\}} \right) \left( U_1^\top M_{u_*,H} \right)^{-1}. \]

Thus, the \( r \) roots of the polynomial \( \lambda \mapsto \det(\bar{C}(m) - \lambda I) \) are \( \{\langle m, M_{(a,b);H,\{S;k\}} e_j \rangle : j \in [r]\} \). We now have show that \( M \) and \( \bar{M} \) are close under correlation decay.
Proposition 5 (Regime of Correlation Decay) For all $k \in \mathcal{Y}^{|S|}$ and $q \in \mathcal{Y}^2$, we have
\[
\|\tilde{M}_{u_*,c,\{S;k\},\{(a,b);q\}} - M_{u_*,c,\{S;k\},\{(a,b);q\}}\|_2 \leq \zeta(\gamma),
\]
where $\zeta(\gamma)$ is given by \((20)\).

Proof: On lines of obtaining \((75)\) in the proof of Lemma \([11]\) it is easy to see that
\[
\|P(Y_c|Y_S = k, Y_{a,b} = q) - \sum_{h \in [r]} P(Y_c|Y_S = k, H = h)P(H = h|Y_S = k, Y_{a,b} = q)\|_1 \leq 2 \max_{h \in [r]} \zeta_h(\gamma).
\]
This implies that for all $y \in \mathcal{Y}$,
\[
\|\sum_{h \in [r]} P(Y_{u_*} = y|H = h)P(H = h, Y_S = k, Y_{a,b} = q)P(Y_c|Y_S = k, H = h) - P(Y_c, Y_{u_*} = y, Y_S = k, Y_{a,b} = q)\|_1 \leq 2 \max_{h \in [r]} \zeta_h(\gamma).
\]
This is the same as
\[
\|\tilde{M}_{u_*,c,\{S;k\},\{(a,b);q\}} - M_{u_*,c,\{S;k\},\{(a,b);q\}}\|_2 \leq 2 \max_{h \in [r]} \zeta_h(\gamma),
\]
where $\|A\|_\infty$ is the maximum absolute row sum and $\|A\|_2 \leq \sqrt{\det}|A|_\infty$ for a $d \times d$ matrix, and thus, we have the result. \(\square\)

E.3 Spectral Bounds under Local Separation

The result follows on similar lines as Section \([D.3]\) except that the distortion between the sample version of the observable operator $\tilde{C}(\textbf{m})$ and the desired version $\tilde{C}(\textbf{m})$ changes. This leads to a slightly different bound

Lemma 12 (Bounds for $\|\tilde{M}_{a,b|H,\{S,k\}}\textbf{e}_j - M_{a,b|H,\{S,k\}}\textbf{e}_{\tau(j)}\|_2$) For any $a, b \in V \setminus \{u_*\}, k \in \mathcal{Y}^{|S|}$, $j \in [r]$, there exists a permutation $\tau(j) \in [r]$ such that, conditioned on event that $\tilde{G}_U = G_U$, with probability at least $1 - 3\delta$,
\[
\|\tilde{M}_{a,b|H,\{S,k\}}\textbf{e}_j - M_{a,b|H,\{S,k\}}\textbf{e}_{\tau(j)}\|_2 \leq \frac{K(\delta; p, d, r)}{\sqrt{n}} + K'(\delta; p, d, r)\zeta(\gamma),
\]
where $K'$ and $K$ are given by \((57)\) and \((58)\), and $\zeta(\gamma)$ is given by \((20)\). This implies
\[
\|\tilde{M}_{a,b|H}\textbf{e}_j - M_{a,b|H}\textbf{e}_{\tau(j)}\|_2 \leq \frac{2K(\delta; p, d, r)}{\sqrt{n}} + 2K'(\delta; p, d, r)\zeta(\gamma).
\]

F Matrix perturbation analysis

We borrow the following results on matrix perturbation bounds from \([Anandkumar et al., 2012b]\). We denote the $p$-norm of a vector $\vec{v}$ by $\|\vec{v}\|_p$, and the corresponding induced norm of a matrix $A$ by $\|A\|_p := \sup_{\vec{v} \neq \vec{0}}\|A\vec{v}\|_p/\|\vec{v}\|_p$. The Frobenius norm of a matrix $A$ is denoted by $\|A\|_F$. For a matrix $A \in \mathbb{R}^{m \times n}$, let $\kappa(A) := \sigma_1(A)/\sigma_{\text{min}(m,n)}(A)$ (thus $\kappa(A) = \|A\|_2 \cdot \|A^{-1}\|_2$ if $A$ is invertible).
Lemma 13 Let $X \in \mathbb{R}^{m \times n}$ be a matrix of rank $k$. Let $U \in \mathbb{R}^{m \times k}$ and $V \in \mathbb{R}^{n \times k}$ be matrices with orthonormal columns such that $\text{range}(U)$ and $\text{range}(V)$ are spanned by, respectively, the left and right singular vectors of $X$ corresponding to its $k$ largest singular values. Similarly define $\hat{U} \in \mathbb{R}^{m \times k}$ and $\hat{V} \in \mathbb{R}^{n \times k}$ relative to a matrix $\hat{X} \in \mathbb{R}^{m \times n}$. Define $\epsilon_X := \|\hat{X} - X\|_2$, $\epsilon_0 := \frac{\epsilon}{\sigma_k(X)}$, and $\epsilon_1 := \frac{\epsilon_0}{1 + \epsilon_0}$. Assume $\epsilon_0 < \frac{1}{2}$. Then

1. $\epsilon_1 < 1$; 
2. $\sigma_k(\hat{X}) = \sigma_k(\hat{U}^\top \hat{X} \hat{V}) \geq (1 - \epsilon_0) \cdot \sigma_k(X) > 0$; 
3. $\sigma_k(\hat{U}^\top U) \geq \sqrt{1 - \epsilon_1^2}$; 
4. $\sigma_k(\hat{V}^\top V) \geq \sqrt{1 - \epsilon_1^2}$; 
5. $\sigma_k(\hat{U}^\top \hat{X} \hat{V}) \geq (1 - \epsilon_1^2) \cdot \sigma_k(X)$; 
6. for any $\alpha \in \mathbb{R}^k$ and $\nu \in \text{range}(U)$, $\|\hat{U} \alpha - \nu\|_2^2 \leq \|\hat{\alpha} - \hat{U}^\top \nu\|_2^2 + \|\nu\|_2^2 \cdot \epsilon_1^2$.

Lemma 14 Consider the setting and definitions from Lemma 13 and let $Y \in \mathbb{R}^{m \times n}$ and $\hat{Y} \in \mathbb{R}^{m \times n}$ be given. Define $\epsilon_2 := \frac{\epsilon_0}{(1 - \epsilon_1)(1 - \epsilon_0 - \epsilon_1^2)}$ and $\epsilon_Y := \|\hat{Y} - Y\|_2$. Assume $\epsilon_0 < \frac{1}{1 + \sqrt{2}}$. Then

1. $\hat{U}^\top \hat{X} \hat{V}$ and $\hat{U}^\top \hat{X} \hat{V}$ are both invertible, and $\| (\hat{U}^\top \hat{X} \hat{V})^{-1} - (\hat{U}^\top \hat{X} \hat{V})^{-1} \|_2 \leq \frac{\epsilon_2}{\sigma_k(X)}$; 
2. $\| (\hat{U}^\top \hat{Y} \hat{V})(\hat{U}^\top \hat{X} \hat{V})^{-1} - (\hat{U}^\top \hat{Y} \hat{V})(\hat{U}^\top \hat{X} \hat{V})^{-1} \|_2 \leq \frac{\epsilon_2}{\sigma_k(X)} + \frac{\|Y\|_2 \epsilon_3}{\sigma_k(X)}$.

Lemma 15 Let $A \in \mathbb{R}^{k \times k}$ be a diagonalizable matrix with $k$ distinct real eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k \in \mathbb{R}$ corresponding to the (right) eigenvectors $\hat{\xi}_1, \hat{\xi}_2, \ldots, \hat{\xi}_k \in \mathbb{R}^k$ all normalized to have $\|\hat{\xi}_i\|_2 = 1$. Let $R \in \mathbb{R}^{k \times k}$ be the matrix whose $i^{th}$ column is $\hat{\xi}_i$. Let $\hat{A} \in \mathbb{R}^{k \times k}$ be a matrix. Define $\epsilon_A := \|\hat{A} - A\|_2$, $\gamma_A := \min_{i \neq j} |\lambda_i - \lambda_j|$, and $\epsilon_3 := \frac{k(R) \epsilon_A}{\gamma_A}$. Assume $\epsilon_3 < \frac{1}{2}$. Then there exists a permutation $\tau$ on $[k]$ such that the following holds:

1. $\hat{A}$ has $k$ distinct real eigenvalues $\hat{\lambda}_1, \hat{\lambda}_2, \ldots, \hat{\lambda}_k \in \mathbb{R}$, and $|\hat{\lambda}_{\tau(i)} - \hat{\lambda}_i| \leq \epsilon_3 \cdot \gamma_A$ for all $i \in [k]$; 
2. $\hat{A}$ has corresponding (right) eigenvectors $\hat{\xi}_1, \hat{\xi}_2, \ldots, \hat{\xi}_k \in \mathbb{R}^k$, normalized to have $\|\hat{\xi}_i\|_2 = 1$, which satisfy $\|\hat{\xi}_{\tau(i)} - \hat{\xi}_i\|_2 \leq 4(k - 1) \cdot \|R^{-1}\|_2 \cdot \epsilon_3$ for all $i \in [k]$; 
3. the matrix $\hat{R} \in \mathbb{R}^{k \times k}$ whose $i^{th}$ column is $\hat{\xi}_{\tau(i)}$ satisfies $\|\hat{R} - R\|_2 \leq \|\hat{R} - R\|_F \leq 4k^{1/2}(k - 1) \cdot \|R^{-1}\|_2 \cdot \epsilon_3$.

Lemma 16 Let $A_1, A_2, \ldots, A_k \in \mathbb{R}^{k \times k}$ be diagonalizable matrices that are diagonalized by the same matrix invertible $R \in \mathbb{R}^{k \times k}$ with unit length columns $\|R_{i,j}\|_2 = 1$, such that each $A_i$ has $k$ distinct real eigenvalues:

$$R^{-1} A_i R = \text{Diag}(\lambda_{i,1}, \lambda_{i,2}, \ldots, \lambda_{i,k}).$$

Let $\hat{A}_1, \hat{A}_2, \ldots, \hat{A}_k \in \mathbb{R}^{k \times k}$ be given. Define $\epsilon_A := \max_i \|\hat{A}_i - A_i\|_2$, $\gamma_A := \min_{i \neq j} \min_{k \neq j'} |\lambda_{i,j} - \lambda_{i,j'}|$, $\lambda_{\max} := \max_{i,j} |\lambda_{i,j}|$, $\epsilon_3 := \frac{k(R) \epsilon_A}{\gamma_A}$, and $\epsilon_4 := 4k^{1.5} \cdot \|R^{-1}\|_2 \cdot \epsilon_3$. Assume $\epsilon_3 < \frac{1}{4}$ and $\epsilon_4 < 1$. Then there exists a permutation $\tau$ on $[k]$ such that the following holds.
Lemma 17 Let $A_1$ has $k$ distinct real eigenvalues $\lambda_{1,1}, \lambda_{1,2}, \ldots, \lambda_{1,k} \in \mathbb{R}$, and $|\lambda_{1,j} - \lambda_{1,\tau(j)}| \leq \varepsilon_3 \cdot \gamma_A$ for all $j \in [k]$.

2. There exists a matrix $\hat{R} \in \mathbb{R}^{k \times k}$ whose $j^{th}$ column is a right eigenvector corresponding to $\hat{\lambda}_{1,j}$, scaled so $\|\hat{R}\hat{e}_j\|_2 = 1$ for all $j \in [k]$, such that $\|\hat{R} - R_\tau\|_2 \leq \frac{\varepsilon_4}{\|\lambda_j\|_2}$, where $R_\tau$ is the matrix obtained by permuting the columns of $R$ with $\tau$.

3. The matrix $\hat{R}$ is invertible and its inverse satisfies $\|\hat{R}^{-1} - R_\tau^{-1}\|_2 \leq \|R_\tau^{-1}\|_2 \cdot \frac{\varepsilon_4}{1 - \varepsilon_4}$.

4. For all $i \in \{2, 3, \ldots, k\}$ and all $j \in [k]$, the $(j, j)^{th}$ element of $\hat{R}^{-1}A_i\hat{R}$, denoted by $\hat{\lambda}_{i,j} := e_j^\top \hat{R}^{-1}A_i\hat{R}e_j$, satisfies

$$|\hat{\lambda}_{i,j} - \lambda_{i,\tau(j)}| \leq \left(1 + \frac{\varepsilon_4}{1 - \varepsilon_4}\right) \cdot \left(1 + \frac{\varepsilon_4}{\sqrt{k} \cdot \kappa(R)}\right) \cdot \varepsilon_3 \cdot \gamma_A + \kappa(R) \cdot \left(\frac{1}{1 - \varepsilon_4} + \frac{1}{\sqrt{k} \cdot \kappa(R)} + \frac{\varepsilon_4}{1 - \varepsilon_4}\right) \cdot \varepsilon_4 \cdot \lambda_{\max}.$$  

If $\varepsilon_4 \leq \frac{1}{2}$, then $|\hat{\lambda}_{i,j} - \lambda_{i,\tau(j)}| \leq 3\varepsilon_3 \cdot \gamma_A + 4\kappa(R) \cdot \varepsilon_4 \cdot \lambda_{\max}$.

**Lemma 17** Let $V \in \mathbb{R}^{k \times k}$ be an invertible matrix, and let $R \in \mathbb{R}^{k \times k}$ be the matrix whose $j^{th}$ column is $Ve_j/\|Ve_j\|_2$. Then $\|R\|_2 \leq \kappa(V)$, $\|R^{-1}\|_2 \leq \kappa(V)$, and $\kappa(R) \leq \kappa(V)^2$.

**References**


