Super-Resolution Off the Grid

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Abstract

Super-resolution is the problem of recovering a superposition of point sources using bandlimited measurements, which may be corrupted with noise. This signal processing problem arises in numerous imaging problems, ranging from astronomy to biology to spectroscopy, where it is common to take (coarse) Fourier measurements of an object. Of particular interest is in obtaining estimation procedures which are robust to noise, with the following desirable statistical and computational properties: we seek to use coarse Fourier measurements (bounded by some cutoff frequency); we hope to take a (quantifiably) small number of measurements; we desire our algorithm to run quickly.

Suppose we have $k$ point sources in $d$ dimensions, where the points are separated by at least $\Delta$ from each other (in Euclidean distance). This work provides an algorithm with the following favorable guarantees:

- The algorithm uses Fourier measurements, whose frequencies are bounded by $O(1/\Delta)$ (up to log factors). Previous algorithms require a cutoff frequency which may be as large as $\Omega(\sqrt{d}/\Delta)$.
- The number of measurements taken by and the computational complexity of our algorithm are bounded by a polynomial in both the number of points $k$ and the dimension $d$, with no dependence on the separation $\Delta$. In contrast, previous algorithms depended inverse polynomially on the minimal separation and exponentially on the dimension for both of these quantities.

Our estimation procedure itself is simple: we take random bandlimited measurements (as opposed to taking an exponential number of measurements on the hyper-grid). Furthermore, our analysis and algorithm are elementary (based on concentration bounds for sampling and the singular value decomposition).
1 Introduction

We follow the standard mathematical abstraction of this problem (Candes & Fernandez-Granda \[4, 3\]): consider a $d$-dimensional signal $x(t)$ modeled as a weighted sum of $k$ Dirac measures in $\mathbb{R}^d$:

$$x(t) = \sum_{j=1}^{k} w_j \delta_{\mu(j)},$$

where the point sources, the $\mu(j)$'s, are in $\mathbb{R}^d$. Assume that the weights $w_j$ are complex valued, whose absolute values are lower and upper bounded by some positive constant. Assume that we are given $k$, the number of point sources.

Define the measurement function $f(s) : \mathbb{R}^d \to \mathbb{C}$ to be the convolution of the point source $x(t)$ with a low-pass point spread function $e^{i\pi<s,t>}$ as below:

$$f(s) = \int_{t \in \mathbb{R}^d} e^{i\pi<t,s>} x(dt) = \sum_{j=1}^{k} w_j e^{i\pi<\mu(j),s>}.$$  \hspace{1cm} (2)

In the noisy setting, the measurements are corrupted by uniformly bounded perturbation $z$:

$$\tilde{f}(s) = f(s) + z(s), \quad |z(s)| \leq \epsilon_z, \forall s.$$  \hspace{1cm} (3)

Suppose that we are only allowed to measure the signal $x(t)$ by evaluating the measurement function $\tilde{f}(s)$ at any $s \in \mathbb{R}^d$, and we want to recover the parameters of the point source signal, i.e., $\{w_j, \mu(j) : j \in [k]\}$. We follow the standard normalization to assume that:

$$\mu(j) \in [-1, +1]^d, \quad |w_j| \in [0, 1] \quad \forall j \in [k].$$

Let $w_{min} = \min_j |w_j|$ denote the minimal weight, and let $\Delta$ be the minimal separation of the point sources defined as follows:

$$\Delta = \min_{j \neq j'} \|\mu(j) - \mu(j')\|_2,$$  \hspace{1cm} (4)

where we use the Euclidean distance between the point sources for ease of exposition. These quantities are key parameters in our algorithm and analysis. Intuitively, the recovery problem is harder if the minimal separation $\Delta$ is small and the minimal weight $w_{min}$ is small.

The first question is that, given exact measurements, namely $\epsilon_z = 0$, where and how many measurements should we take so that the original signal $x(t)$ can be exactly recovered.

**Definition 1.1 (Exact recovery).** In the exact case, i.e. $\epsilon_z = 0$, we say that an algorithm achieves exact recovery with $m$ measurements of the signal $x(t)$ if, upon input of these $m$ measurements, the algorithm returns the exact set of parameters $\{w_j, \mu(j) : j \in [k]\}$.

Moreover, we want the algorithm to be measurement noise tolerant, in the sense that in the presence of measurement noise we can still recover good estimates of the point sources.

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1 An upper bound of the number of point sources suffices.

2 Our claims hold without using the “wrap around metric”, as in \[4, 3\], due to our random sampling. Also, it is possible to extend these results for the $\ell_p$-norm case.
**Definition 1.2 (Stable recovery).** In the noisy case, i.e., $\epsilon_z \geq 0$, we say that an algorithm achieves stable recovery with $m$ measurements of the signal $x(t)$ if, upon input of these $m$ measurements, the algorithm returns estimates $\{\hat{w}_j, \hat{\mu}^{(j)} : j \in [k]\}$ such that

$$\min_{\pi} \max_{j \in [k]} \left\{ \|\hat{\mu}^{(j)} - \mu^{(\pi(j))}\|_2 : j \in [k] \right\} \leq \text{poly}(d, k)\epsilon_z,$$

where the min is over permutations $\pi$ on $[k]$ and poly$(d,k)$ is a polynomial function in $d$ and $k$.

By definition, if an algorithm achieves stable recovery with $m$ measurements, it also achieves exact recovery with these $m$ measurements.

The terminology of “super-resolution” is appropriate due to the following remarkable result (in the noiseless case) of Donoho [9]: suppose we want to accurately recover the point sources to an error of $\gamma$, where $\gamma \ll \Delta$. Naively, we may expect to require measurements whose frequency depends inversely on the desired the accuracy $\gamma$. Donoho [9] showed that it suffices to obtain a finite number of measurements, whose frequencies are bounded by $O(1/\Delta)$, in order to achieve exact recovery; thus resolving the point sources far more accurately than that which is naively implied by using frequencies of $O(1/\Delta)$. Furthermore, the work of Candes & Fernandez-Granda [4, 3] showed that stable recovery, in the univariate case ($d = 1$), is achievable with a cutoff frequency of $O(1/\Delta)$ using a convex program and a number of measurements whose size is polynomial in the relevant quantities.

**1.1 This work**

We are interested in stable recovery procedures with the following desirable statistical and computational properties: we seek to use coarse (low frequency) measurements; we hope to take a (quantifiably) small number of measurements; we desire our algorithm run quickly. Informally, our main result is as follows:

**Theorem 1.3 (Informal statement of Theorem 3.2).** For a fixed probability of error, the proposed algorithm achieves stable recovery with a number of measurements and with computational runtime that are both on the order of $O((k \log(k) + d)^2)$. Furthermore, the algorithm makes measurements which are bounded in frequency by $O(1/\Delta)$ (ignoring log factors).

Notably, our algorithm and analysis directly deal with the multivariate case, with the univariate case as a special case. Importantly, the number of measurements and the computational runtime do not depend on the minimal separation of the point sources. This may be important even in certain low dimensional imaging applications where taking physical measurements are costly (indeed, super-resolution is important in settings where $\Delta$ is small). Furthermore, our technical contribution of how to decompose a certain tensor constructed with Fourier measurements may be of broader interest to related questions in statistics, signal processing, and machine learning.

**1.2 Comparison to related work**

Table 1 summarizes the comparisons between our algorithm and the existing results. The multidimensional cutoff frequency we refer to in the table is the maximal coordinate-wise entry of any measurement frequency $s$ (i.e. $\|s\|_\infty$). “SDP” refers to the semidefinite programming (SDP) based algorithms of Candes & Fernandez-Granda [3 4]; in the univariate case, the number of measurements can be reduced by the method in Tang et. al. [23] (this is reflected in the table). “MP” refers to the matrix pencil type of methods, studied in [14] and [15] for the univariate case. Here, we are defining the infinity norm separation as $\Delta_\infty = \min_{j \neq j'} \|\mu^{(j)} - \mu^{(j')}\|_\infty$, which is understood
as the wrap around distance on the unit circle. $C_d \geq 1$ is a problem dependent constant (discussed below).

Observe the following differences between our algorithm and prior work:

1) Our minimal separation is measured under the $\ell_2$-norm instead of the infinity norm, as in the SDP based algorithm. Note that $\Delta_{\infty}$ depends on the coordinate system; in the worst case, it can underestimate the separation by a $1/\sqrt{d}$ factor, namely $\Delta_{\infty} \sim \Delta/\sqrt{d}$.

2) The computation complexity and number of measurements are polynomial in dimension $d$ and the number of point sources $k$, and surprisingly do not depend on the minimal separation of the point sources! Intuitively, when the minimal separation between the point sources is small, the problem should be harder, this is only reflected in the sampling range and the cutoff frequency of the measurements in our algorithm.

3) Furthermore, one could project the multivariate signal to the coordinates and solve multiple univariate problems (such as in [19, 17], which provided only exact recovery results). Naive random projections would lead to a cutoff frequency of $O(\sqrt{d}/\Delta)$.

### SDP approaches:
The work in [3, 4, 10] formulates the recovery problem as a total-variation minimization problem; they then show the dual problem can be formulated as an SDP. They focused on the analysis of $d = 1$ and only explicitly extend the proofs for $d = 2$. For $d \geq 1$, Ingham-type theorems (see [20, 12]) suggest that $C_d = O(\sqrt{d})$.

The number of measurements can be reduced by the method in [23] for the $d = 1$ case, which is noted in the table. Their method uses sampling “off the grid”; technically, their sampling scheme is actually sampling random points from the grid, though with far fewer measurements.

### Matrix pencil approaches:
The matrix pencil method, MUSIC and Prony’s method are essentially the same underlying idea, executed in different ways. The original Prony’s method directly attempts to find roots of a high degree polynomial, where the root stability has few guarantees. Other methods aim to robustify the algorithm.

Recently, for the univariate matrix pencil method, Liao & Fannjiang [14] and Moitra [15] provide a stability analysis of the MUSIC algorithm. Moitra [15] studied the optimal relationship between the cutoff frequency and $\Delta$, showing that if the cutoff frequency is less than $1/\Delta$, then stable recovery is not possible with matrix pencil method (with high probability).

### 1.3 Notation

Let $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{Z}$ to denote real, complex, and natural numbers. For $d \in \mathbb{Z}$, $[d]$ denotes the set $[d] = \{1, \ldots, d\}$. For a set $S$, $|S|$ denotes its cardinality. We use $\oplus$ to denote the direct sum of sets.
The Vandermonde matrix \( V \)

Let \( e_n \) denote the \( n \)-th standard basis vector in \( \mathbb{R}^d \), for \( n \in [d] \). Let \( \mathcal{P}^d_{R,2} = \{ x \in \mathbb{R}^d : \|x\|_2 = 1 \} \) to denote the \( d \)-sphere of radius \( R \) in the \( d \)-dimensional standard Euclidean space.

Denote the condition number of a matrix \( X \in \mathbb{R}^{m \times n} \) as \( \text{cond}_2(X) = \sigma_{\text{max}}(X)/\sigma_{\text{min}}(X) \), where \( \sigma_{\text{max}}(X) \) and \( \sigma_{\text{min}}(X) \) are the maximal and minimal singular values of \( X \).

We use \( \otimes \) to denote tensor product. Given matrices \( A, B, C \in \mathbb{C}^{m \times k} \), the tensor product \( V = A \otimes B \otimes C \in \mathbb{C}^{m \times m \times m} \) is equivalent to \( V_{i_1,i_2,i_3} = \sum_{n=1}^{k} A_{i_1,n}B_{i_2,n}C_{i_3,n} \). Another view of tensor is that it defines a multi-linear mapping. For given dimension \( m_A, m_B, m_C \) the mapping \( V(\cdot,\cdot,\cdot) : \mathbb{C}^{m_A \times m_B} \times \mathbb{C}^{m_B \times m_C} \rightarrow \mathbb{C}^{m_A \times m_B \times m_C} \) is defined as:

\[
[V(X_A,X_B,X_C)]_{i_1,i_2,i_3} = \sum_{j_1,j_2,j_3 \in [m]} V_{j_1,j_2,j_3}[X_A]_{j_1,i_1}[X_B]_{j_2,i_2}[X_C]_{j_3,i_3}.
\]

In particular, for \( a \in \mathbb{C}^m \), we use \( V(I,I,a) \) to denote the projection of tensor \( V \) along the 3rd dimension. Note that if the tensor admits a decomposition \( V = A \otimes B \otimes C \), it is straightforward to verify that

\[
V(I,I,a) = AD_{\text{diag}}(C^T a) B^T.
\]

It is well-known that if the factors \( A, B, C \) have full column rank then the rank \( k \) decomposition is unique up to re-scaling and common column permutation. Moreover, if the condition number of the factors are upper bounded by a positive constant, then one can compute the unique tensor decomposition \( V \) with stability guarantees (See [1] for a review. Lemma 3.5 herein provides an explicit statement.).

2 Warm-up

2.1 1-D case: revisiting the matrix pencil method

Let us first review the matrix pencil method for the univariate case, which stability was recently rigorously analyzed in Liao & Fannjiang [14] and Moitra [15].

A square matrix \( H \) is called a Hankel matrix if its skew-diagonals are constants, namely \( H_{i,j} = H_{i-1,j+1} \). For some positive constants \( m \in \mathbb{Z} \), sample to get the measurements \( f(s) \) evaluated at the sampling set \( \mathcal{S}_3 = \{0,1,\ldots,2m\} \), and construct two Hankel matrices \( H_0, H_1 \in \mathbb{C}^{m \times m} \):

\[
H_0 = \begin{bmatrix}
    f(0) & f(1) & \ldots & f(m-1) \\
    f(1) & f(2) & \ldots & f(m) \\
    \vdots & \vdots & \ddots & \vdots \\
    f(m-1) & f(m) & \ldots & f(2m-1)
\end{bmatrix}, \quad H_1 = \begin{bmatrix}
    f(1) & f(2) & \ldots & f(m) \\
    f(2) & f(3) & \ldots & f(m+1) \\
    \vdots & \vdots & \ddots & \vdots \\
    f(m) & f(m+1) & \ldots & f(2m)
\end{bmatrix}. \tag{5}
\]

Define \( D_w \in \mathbb{C}^{k \times k}_{\text{diag}} \) to be the diagonal matrix with the weights on the main diagonal: \([D_w]_{j,j} = w_j \). Define \( D_\mu \in \mathbb{C}^{k \times k}_{\text{diag}} \) to be \([D_\mu]_{j,j} = e^{i\pi \mu(j)} \).

A matrix \( V \) is called a Vandermonde matrix if each column is a geometric progression. defined the Vandermonde matrix \( V_m \in \mathbb{C}^{m \times k} \) as below:

\[
V_m = \begin{bmatrix}
    1 & \ldots & 1 \\
    (e^{i\pi \mu(1)})^1 & \ldots & (e^{i\pi \mu(k)})^1 \\
    \vdots & \ddots & \vdots \\
    (e^{i\pi \mu(1)})^{m-1} & \ldots & (e^{i\pi \mu(k)})^{m-1}
\end{bmatrix}. \tag{6}
\]
The two Hankel matrices $H_0$ and $H_1$ admit the following simultaneous diagonalization:

$$H_0 = V_m D_w V_m^\top, \quad H_1 = V_m D_\mu V_m^\top.$$  \hfill (7)

As long as $V_m$ is of full rank, this simultaneous diagonalization can be computed by solving the generalized eigenvalue problem, and the parameters of the point source can thus be obtained from the factor $V_m$ and $D_w$.

The univariate matrix pencil method only needs $m \geq k$ to achieve exact recovery. In the noisy case, the stability of generalized eigenvalue problem depends on the condition number of the Vandermonde matrix $V_m$ and the minimal weight $w_{\text{min}}$.

Since all the nodes $(e^{i\pi \mu^{(j)}})s$ of this Vandermonde matrix lie on the unit circle in the complex plane, it is straightforward to see that asymptotically $\lim_{m \to \infty} \text{cond}_2(V_m) = 1$. Furthermore, for $m > 1/\Delta$, [14, 15] showed that $\text{cond}_2(V_m)$ is upper bounded by a constant that does not depend on $k$ and $m$. This bound on condition number is also implicitly discussed in [10].

Another way to view the matrix pencil method is that it corresponds to the low rank 3rd order tensor decomposition (see for example [11]). This view will help us generalize matrix pencil method to higher dimension $d$ in a direct way, without projecting the signal on each coordinate and apply the univariate algorithm multiple times. For $m \geq k$, construct a 3rd order tensor $F \in \mathbb{C}^{m \times m \times 2}$ with elements of $H_0$ and $H_1$ defined in (5) as:

$$F_{i,i',j} = [H_{j-1}]_{i,i'}, \quad \forall j \in [2], i, i' \in [m].$$

Note that the two slices along the 3rd dimension of $F$ are $H_0$ and $H_1$. Namely $F(I,I,e_1) = H_0$, and $F(I,I,e_2) = H_1$. Recall the matrix decomposition of $H_0$ and $H_1$ in (7). Since $m \geq k$ and the $\mu^{(j)}$s are distinct, we know that $F$ has the unique rank $k$ tensor decomposition:

$$F = V_m \otimes V_m \otimes (V_2 D_w).$$

Given the tensor $F$, the basic idea of the well-known Jennrich’s algorithm ([11, 13]) for finding the unique low rank tensor decomposition is to consider two random projections $v_1, v_2 \in \mathbb{R}^m$, and then with high probability the two matrices $F(I,I,v_1)$ and $F(I,I,v_2)$ admit simultaneous diagonalization. Therefore, the matrix pencil method is indeed a special case of Jennrich’s algorithm by setting $v_1 = e_1$ and $v_2 = e_2$.

### 2.2 The multivariate case: a toy example

One could naively extend the matrix pencil method to higher dimensions by using taking measurements from a hyper-grid, which is of size exponential in the dimension $d$. We now examine a toy problem which suggests that the high dimensional case may not be inherently more difficult than the univariate case.

The key idea is that an appropriately sampled set can significantly reduce the number of measurements (as compared to using all the grid points). Tang et al [23] made a similar observation for the univariate case. They used a small random subset of measurements (actually still from the grid points) and showed that this contains enough information to recover all the measurement on the grid; the full measurements were then used for stably recovering the point sources.

Consider the case where the dimension $d \geq k$. Assume that $w_j$’s are real valued, and for all $j \in [k]$ and $n \in [d]$, the parameters $\mu^{(j)}_n$ are i.i.d. and uniformly distributed over $[-1, +1]$. This essentially corresponds to the standard $(L_2)$ incoherence conditions (for the $\mu^{(j)}$’s).

The following simple algorithm achieves stability with polynomial complexity.

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4 This setting is different from the 2-norm separation condition. To see the difference, note that the toy algorithm does not work for constant shift $\mu^{(1)} = \mu^{(2)} + \Delta$. This issue is resolved in the general algorithm, when the condition is stated in terms of 2-norm separation.
First, take \(d^3\) number of measurements by evaluating \(f(s)\) in the set \(\mathcal{S}_3 = \{s = e_{n_1} + e_{n_2} + e_{n_3} : [n_1, n_2, n_3] \in [d] \times [d] \times [d]\}\), noting that \(\mathcal{S}_3\) contains only a subset of \(d^3\) points from the grid of \([3]^d\). Then, construct a 3rd order tensor \(F \in \mathbb{C}^{[d]} \times [d] \times [d]\) with the measurements in the following way:

\[
F_{n_1,n_2,n_3} = f(s) \big|_{s = e_{n_1} + e_{n_2} + e_{n_3}}, \quad \forall n_1,n_2,n_3 \in [d].
\]

Note that the measurement \(f(e_1 + e_2 + e_3) = \sum_{j=1}^{k} w_je^{i\pi(\mu_1^{(j)} + \mu_2^{(j)} + \mu_3^{(j)})} = \sum_{j=1}^{k} w_je^{i\pi\mu_1^{(j)} e^{i\pi\mu_2^{(j)} e^{i\pi\mu_3^{(j)}}}}\). It is straightforward to verify that \(F\) has a rank-\(k\) tensor factorization \(F = V_d \odot V_d \odot (V_d D_w)\), where the factor \(V_d \in \mathbb{R}^{d \times k}\) is given by:

\[
V_d = \begin{bmatrix}
e^{i\pi\mu_1^{(1)}} & \cdots & e^{i\pi\mu_k^{(1)}} \\
e^{i\pi\mu_2^{(1)}} & \cdots & e^{i\pi\mu_k^{(2)}} \\
& \cdots & \\
e^{i\pi\mu_1^{(d)}} & \cdots & e^{i\pi\mu_k^{(d)}}
\end{bmatrix}.
\] (8)

Under the distribution assumption of the point sources, the entries \(e^{i\pi\mu_n^{(j)}}\) are i.i.d. and uniformly distributed over the unit circle on the complex plane. Therefore almost surely the factor \(V_d\) has full column rank, and thus the tensor decomposition is unique. Moreover here \(w_j\)’s are real and each element of \(V_S\) has unit norm, we have a rescaling constraint with the tensor decomposition, with which we can uniquely obtain the factor \(V_S\) and the weights in \(D_w\). By taking element-wise log of \(V_S\) we can read off the parameters of the point sources from \(V_S\) directly. Moreover, with high probability, we have that \(\text{cond}_2(V_d)\) concentrates around 1, thus the simple algorithm achieves stable recovery.

### 3. Main Results

#### 3.1. The algorithm

We briefly describe the steps of Algorithm [1] below:

**Take measurements** Given positive numbers \(m\) and \(R\), randomly draw a sampling set \(\mathcal{S} = \{s^{(1)}, \ldots, s^{(m)}\}\) of \(m\) i.i.d. samples of the Gaussian distribution \(\mathcal{N}(0, R^2 I_{[d] \times [d]}\). Form the set \(\mathcal{S}' = \mathcal{S}\cup\{s^{(m+1)} = e_1, \ldots, s^{(m+d)} = e_d, s^{-(m+d+1)} = 0\} \subset \mathbb{R}^d\). Denote \(m' = m + d + 1\). Take another independent random sample \(v\) from the unit sphere, and define \(v^{(1)} = v, v^{(2)} = 2v\). Construct the 3rd order tensor \(\tilde{F} \in \mathbb{C}^{m' \times m' \times 3}\) with noise corrupted measurements \(\tilde{f}(s)\) evaluated at the points in \(\mathcal{S}' \odot \mathcal{S}' \odot \{v^{(1)}, v^{(2)}\}\), arranged in the following way:

\[
\tilde{F}_{n_1,n_2,n_3} = \tilde{f}(s) \big|_{s = s^{(n_1)} + s^{(n_2)} + v^{(n_3)}}, \quad \forall n_1, n_2 \in [m'] , n_3 \in [2].
\] (9)

**Tensor decomposition** Define the characteristic matrix \(V_S\) to be:

\[
V_S = \begin{bmatrix}
e^{i\pi<\mu^{(1),s^{(1)}}>} & \cdots & e^{i\pi<\mu^{(k),s^{(1)}}>}
\end{bmatrix}.
\] (10)
Input: $R$, $m$, noisy measurement function $\tilde{f}(\cdot)$.
Output: Estimates $\{\hat{w}_j, \hat{\mu}^{(j)} : j \in [k]\}$.

1. Take measurements:
   Let $S = \{s^{(1)}, \ldots, s^{(m)}\}$ be $m$ i.i.d. samples from the Gaussian distribution $\mathcal{N}(0, R^2 I_{d \times d})$.
   Set $s^{(m+n)} = e_n$ for all $n \in [d]$ and $s^{(m+n+1)} = 0$. Denote $m' = m + d + 1$.
   Take another random samples $v$ from the unit sphere, and set $v^{(1)} = v$ and $v^{(2)} = 2v$.
   Construct a tensor $\tilde{F} \in \mathbb{C}^{m' \times m' \times 3}$: $\tilde{F}_{n_1, n_2, n_3} = \tilde{f}(s) \big|_{s = s^{(n_1)} + s^{(n_2)} + v^{(n_3)}}$.

2. Tensor Decomposition: Set $(\hat{V}'_S, \hat{D}_w) = \text{TensorDecomp}(\tilde{F})$.
   For $j = 1, \ldots, k$, set $[\hat{V}'_S]_{j} = [\hat{V}'_S]_{j}/[\hat{V}'_S]_{m',j}$

3. Read of estimates: For $j = 1, \ldots, k$, set $\hat{\mu}^{(j)} = \text{Real}(\log([\hat{V}'_S]_{m+1:m+d,j})/(i\pi))$.

4. Set $\hat{W} = \arg \min_{W \in \mathbb{C}^k} \|\tilde{F} - \hat{V}'_S \otimes \hat{V}'_S \otimes \hat{D}_w\|_F$.

Algorithm 1: General algorithm

and define matrix $V' \in \mathbb{C}^{m' \times k}$ to be

$$V'_S = \begin{bmatrix} V' \left[ \begin{array}{c} V'_S \\ V_d \\ 1, \ldots, 1 \end{array} \right], \right] \tag{11}$$

where $V_d \in \mathbb{C}^{d \times k}$ is defined in (8). Define

$$V_2 = \begin{bmatrix} e^{i\pi \mu^{(1)}}, v^{(1)} > & \cdots & e^{i\pi \mu^{(k)}, v^{(1)} >} \\
 e^{i\pi \mu^{(1)}, v^{(2)} >} & \cdots & e^{i\pi \mu^{(k)}, v^{(2)} >} \\
 1 & \cdots & 1 \end{bmatrix}.$$

Note that in the exact case ($\epsilon_z = 0$) the tensor $F$ constructed in (9) admits a rank-$k$ decomposition:

$$F = V'_S \otimes V'_S \otimes (V_2 D_w), \tag{12}$$

Assume that $V'_S$ has full column rank, then this tensor decomposition is unique up to column permutation and rescaling with very high probability over the randomness of the random unit vector $v$. Since each element of $V'_S$ has unit norm, and we know that the last row of $V'_S$ and the last row of $V_2$ are all ones, there exists a proper scaling so that we can uniquely recover $w_j$’s and columns of $V'_S$ up to common permutation.

In this paper, we adopt Jennrich’s algorithm (see Algorithm 2) for tensor decomposition. Other algorithms, for example tensor power method (11), and recursive projection (24), which are possibly more stable than Jennrich’s algorithm, can also be applied here.

(Read off estimates) Let $\log(V_d)$ denote the element-wise logarithm of $V_d$. The estimates of the point sources are given by:

$$[\mu^{(1)}, \mu^{(2)}, \ldots, \mu^{(k)}] = \log(V_d) / i\pi.$$
Input: Tensor $\tilde{F} \in \mathbb{C}^{m \times m \times 3}$, rank $k$.

output: Factor $\hat{V} \in \mathbb{C}^{m \times k}$.

1. Compute the truncated SVD of $\tilde{F}(I,I,e_1) = \tilde{P}\tilde{\Lambda}\tilde{P}^\top$ with the $k$ leading singular values.
2. Set $\hat{E} = \tilde{F}(\tilde{P},\tilde{P},I)$. Set $\hat{E}_1 = \hat{E}(I,I,e_1)$ and $\hat{E}_2 = \hat{E}(I,I,e_2)$.
3. Let the columns of $\hat{U}$ be the eigenvectors of $\hat{E}_1\hat{E}_2^{-1}$ corresponding to the $k$ eigenvalues with the largest absolute value.
4. Set $\hat{V} = \sqrt{m}\hat{P}\hat{U}$.

Algorithm 2: TensorDecomp

**Remark 3.1.** In the toy example, the simple algorithm corresponds to using the sampling set $S' = \{e_1, \ldots, e_d\}$. The conventional univariate matrix pencil method corresponds to using the sampling set $S' = \{0, 1, \ldots, m\}$ and the set of measurements $S' \oplus S' \oplus S'$ corresponds to the grid $[m]^3$.

### 3.2 Guarantees

In this section, we discuss how to pick the two parameters $m$ and $R$ and prove that the proposed algorithm indeed achieves stable recovery in the presence of measurement noise.

**Theorem 3.2 (Stable recovery).** There exists a universal constant $C$ such that the following holds.

Fix $\epsilon_x, \delta_x, \delta_v \in (0,\frac{1}{2})$; pick $m$ such that $m \geq \max \left\{ \frac{k}{\epsilon_x} \sqrt{8\log \frac{k}{\delta_x}} \cdot d \right\}$; for $d = 1$, pick $R \geq \frac{\sqrt{2\log(1+2/\epsilon_x)}}{\pi \Delta}$; for $d \geq 2$, pick $R \geq \frac{\sqrt{2\log(k/\epsilon_x)}}{\pi \Delta}$.

Assume the bounded measurement noise model as in (3) and that $\epsilon_x \leq \frac{\Delta_v w_{\min}^2}{100 \sqrt{dk^5 \left( \frac{1-2\epsilon_x}{1+2\epsilon_x} \right)^{2.5}}}$.

With probability at least $(1-\delta_x)$ over the random sampling of $S$, and with probability at least $(1-\delta_v)$ over the random projections in Algorithm 2, the proposed Algorithm 7 returns an estimation of the point source signal $\hat{z}(t) = \sum_{j=1}^{k} \tilde{w}_j \delta_{\mu(j)}$ with accuracy:

$$\min_{\pi} \max_{j \in [k]} \{ \| \hat{\mu}^{(j)} - \mu^{(\pi(j))} \|_2 : j \in [k] \} \leq C \frac{\sqrt{dk^5 w_{\max}}}{\Delta \delta_v} \frac{w_{\min}^2}{1-2\epsilon_x} \epsilon_x^{2.5},$$

where the min is over permutations $\pi$ on $[k]$. Moreover, the proposed algorithm has time complexity in the order of $O((m')^3)$.

**Proof.** (of Theorem 3.2) The algorithm is correct if the tensor decomposition in Step 2 is unique, and achieves stable recovery if the tensor decomposition is stable. By the stability Lemma of tensor decomposition (Lemma 3.5), this is guaranteed if we can bound the condition number of $V_{S'}$. It follows from Lemma 3.8 that the condition number of $V_{S'}$ is at most $\sqrt{1 + \frac{k}{n}}$ times of $\text{cond}_2(V_S)$. By the main technical lemma (Lemma 3.10) we know that with the random sampling set $S$ of size $m$, the condition number $\text{cond}_2(V_S)$ is upper bounded by a constant. Thus we can bound the distance between $V_{S'}$ and the estimation $\hat{V}_{S'}$ according to (13).
Since we adopt Jennrich’s algorithm for the low rank tensor decomposition, the overall computation complexity is roughly the complexity of SVD of a matrix of size \(m' \times m'\), namely in the order of \(O((m')^3)\).

The next lemma shows that essentially, with overwhelming probability, all the frequencies taken concentrate within the hyper-cube with cutoff frequency \(R'\) on each coordinate, where \(R'\) is comparable to \(R\).

**Lemma 3.3** (The cutoff frequency). For \(d > 1\), with high probability, all the \(2(m')^2\) sampling frequencies in \(S' \oplus S' \oplus \{v^{(1)}, v^{(2)}\}\) satisfy that \(|s^{(j_1)} + s^{(j_2)} + v^{(j_3)}| \leq R'\), \(\forall j_1, j_2 \in [m], j_3 \in [2]\), where the per-coordinate cutoff frequency is given by \(R' = O(R\sqrt{\log md})\).

For \(d = 1\) case, the cutoff frequency \(R'\) can be made to be in the order of \(R' = O(1/\Delta)\).

**Proof.** For \(d > 1\) case, with straightforward union bound over the \(m' = O(k^2)\) samples each of which has \(d\) coordinates, one can show that the cutoff frequency is in the order of \(R\sqrt{\log kd}\), where \(R\) is in the order of \(\frac{\sqrt{\log(k)}}{\Delta}\) as shown in Theorem 3.2.

For \(d = 1\) case, we bound the cutoff frequency with slightly more careful analysis. Instead of Gaussian random samples, consider uniform samples from the interval \([-R', R']\). We can modify the proof of Lemma 3.9 and show that if \(R' \geq 1/(\Delta(1 + \epsilon_x))\):

\[
\sum_{j' \neq j} |Y_{j,j'}| = \sum_{j' \neq j} \frac{1}{2R'} \int_{-R',R'} e^{i\pi(\mu(j') - \mu(j))s} \sin(\Delta R') \leq \sum_{j' \neq j} \frac{\sin(\pi \Delta R')}{1 - \sin(\pi \Delta R')/(\pi \Delta R')} \leq \epsilon_x
\]

where the second last inequality uses the inequality that \(\frac{\sin(a+b)}{a+b} \leq \frac{\sin(a)}{a} \leq \frac{\sin(b)}{b}\).

**Remark 3.4** (Failure probability). Overall, the failure probability consists of two pieces: \(\delta_v\) for random projection of \(v\), and \(\delta_s\) for random sampling to ensure the bounded condition number of \(V_S\). This may be boosted to arbitrarily high probability through repetition.

**3.3 Key Lemmas**

**Stability of tensor decomposition:** In this paragraph, we give a brief description and the stability guarantee of the well-known Jennrich’s algorithm \([11, 13]\) for low rank 3rd order tensor decomposition. We only state it for the symmetric tensors as appeared in the proposed algorithm.

Consider a tensor \(F = V \otimes V \otimes (V_2D_w) \in \mathbb{C}^{m \times m \times 3}\) where the factor \(V\) has full column rank \(k\). Then the decomposition is unique up to column permutation and rescaling, and Algorithm 2 finds the factors efficiently. Moreover, the eigen-decomposition is stable if the factor \(V\) is well-conditioned and the eigenvalues of \(F_vF_v^\top\) are well separated.

**Lemma 3.5** (Stability of Jennrich’s algorithm). Consider the 3rd order tensor \(F = V \otimes V \otimes (V_2D_w) \in \mathbb{C}^{m \times m \times 3}\) of rank \(k \leq m\), constructed as in Step 1 in Algorithm 1.

Given a tensor \(\tilde{F}\) that is element-wise close to \(F\), namely for all \(n_1, n_2, n_3 \in [m], |\tilde{F}_{n_1,n_2,n_3} - F_{n_1,n_2,n_3}| \leq \epsilon_x\), and assume that the noise is small \(\epsilon_x \leq \frac{\delta_vw^2_{\text{min}}}{100\delta_vw_{\text{max}}\text{cond}_{2}(V)^5}\). Use \(\tilde{F}\) as the input to Algorithm 2. With probability at least \((1 - \delta_v)\) over the random projections \(v^{(1)}\) and \(v^{(2)}\), we can bound the distance between columns of the output \(\tilde{V}\) and that of \(V\) by:

\[
\min_{\pi} \max_{j} \left\{ \|\tilde{V}_j - V_{\pi(j)}\|_2 : j \in [k] \right\} \leq C\frac{\sqrt{dk^2w_{\text{max}}^2\text{cond}_{2}(V)^5}}{\delta_vw_{\text{min}}^2}\epsilon_z,
\]

\[(13)\]
where $C$ is a universal constant.

**Proof.** (of Lemma 3.5) The proof is mostly based on the arguments in [16, 2], we still show the clean arguments here for our case.

We first introduce some notations for the exact case. Define $D_1 = \text{diag}([V_2]_1, D_w)$ and $D_2 = \text{diag}([V_2]_2, D_w)$. Recall that the symmetric matrix $F_1 = F(I, I, e_1) = V D_1 V^\top$. Consider its SVD $F_1 = P \Delta P^\top$. Denote $U = P^\top V \in \mathbb{C}^{k \times k}$. Define the whitened rank-$k$ tensor $E = F(P, P, I) = (P^\top V) \otimes (P^\top V) \otimes (V_2 D_w) = U \otimes U \otimes (V_2 D_w) \in \mathbb{C}^{k \times k \times 3}$.

Denote the two slices of the tensor $E$ by $E_1 = E(I, I, e_1) = U D_1 U^\top$ and $E_2 = E(I, I, e_2) = U D_2 U^\top$. Define $M = E_1 E_2^{-1}$, and its eigen decomposition is given by $M = U D U^{-1}$, where $D = D_1 D_2^{-1}$. Note that in the exact case, $D$ is given by:

$$D = \text{diag}(e^{i \pi <\mu^{(j)}, \nu^{(1)} - \nu^{(2)}>}: j \in [k])$$

Note that $|D_{j,j}| = 1$ for all $j$. Define the minimal separation of the diagonal entries in $D$ to be:

$$\text{sep}(D) = \min\{\min_{j \neq j'} |D_{j,j} - D_{j',j'}|\},$$

1. We first apply perturbation bounds to show that the noise in $\tilde{F}$ propagates the estimates $\tilde{P}$ and $\tilde{E}$ in a mild way when the condition number of $V$ is bounded by a constant.

**Proof.** Apply Wedin’s matrix perturbation bound, we have:

$$\|\tilde{P} - P\|_2 \leq \frac{\|\tilde{F}_1 - F_1\|}{\sigma_{\min}(F_1)} \leq \frac{\epsilon \sqrt{m}}{\min \sigma_{\min}(V)^2}\]$$

And then for the two slices of $\tilde{E} = \tilde{F}(\tilde{P}, \tilde{P}, I)$, namely $\tilde{E}_i = E_i + Z_i$ for $i = 1, 2$, we can bound the distance between estimates and the exact case, namely $Z_i = \tilde{P}^\top \tilde{F}_i \tilde{P} - P^\top F_i P$, by:

$$\|Z_i\| \leq 8\|F_i\|\|P\|\|\tilde{P} - P\| + 4\|P\|^2\|\tilde{F}_i - F_i\| \leq 16 \frac{\max \sigma_{\min}(V)}{\min \sigma_{\min}(V)} \epsilon \sqrt{m}$$

2. Then, recall that $M = E_1 E_2^{-1} = U D U^{-1}$. Note that

$$\hat{M} = (E_1 + Z_1)(E_2 + Z_2)^{-1} = E_1 E_2^{-1}(I - Z_2(I + E_2^{-1} Z_2)^{-1} E_2^{-1}) + Z_1 E_2^{-1}.$$ 

Let $H$ and $G$ denote the perturbation matrices:

$$H = -Z_2(I + E_2^{-1} Z_2)^{-1} E_2^{-1}, \quad G = Z_1 E_2^{-1}.$$ 

In the following claim, we show that given $\hat{M} = \hat{E}_1 \hat{E}_2^{-1} = M(I + H) + G$ for some small perturbation matrix $H$ and $G$, if the perturbation $\|H\|$ and $\|G\|$ are small enough and that $\text{sep}(D)$ is large enough, the eigen decomposition $\hat{M} = \hat{U} \hat{D} \hat{U}^{-1}$ is close to that of $M$.

**Claim 3.6.** If $\|MH + G\| \leq \frac{\text{sep}(D)}{2 \nu_k \text{cond}_2(U)}$, then the eigenvalues of $\hat{M}$ are distinct and we can bound the columns of $\hat{U}$ and $U$ by:

$$\min \max_{\pi} \|\hat{U}_{\pi(i)} - U_{\pi(i)}\|_2 \leq 3 \frac{\sigma_{\max}(H) \sigma_{\max}(D) + \sigma_{\max}(G)}{\sigma_{\min}(U) \text{sep}(D)} \|\hat{U}_j\|_2 \|V_j\|_2.$$ 

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Proof. Let $\lambda_j$ and $U_j$ for $j \in [k]$ denote the eigenvalue and corresponding eigenvectors of $M$. If $\|MH + G\| \leq \frac{sep(D)}{2 \sqrt{k \text{cond}_2(U)}}$, we can bound

$$
\|\hat{M} - M\| = \|U^{-1}(M + (MH + G))U - D\| = \|U^{-1}(MH + G)U\| \leq \frac{sep(D)}{2\sqrt{k}},
$$

thus apply Gershgorin’s disk theorem, we have $|\hat{\lambda}_j - \lambda_j| \leq \|U^{-1}(MH + G)U\|_1 \leq \sqrt{k}\|U^{-1}(MH + G)U\|_2 \leq \frac{sep(D)}{2}$. Therefore, the eigenvalues are distinct and we have

$$
|\hat{\lambda}_j - \lambda_{j'}| \geq |\lambda_j - \lambda_{j'}| - |\hat{\lambda}_j - \lambda_j| \geq \frac{1}{2} |\lambda_j - \lambda_{j'}| \geq \frac{1}{2} sep(D).
$$

(14)

Note that $\{U_j\}$ and $\{\hat{U}_j\}$ define two sets of basis vectors, thus we can write $\hat{U}_j = \sum_{j'} c_{j'} U_{j'}$ (with the correct permutation for columns of $\hat{U}_j$ and $U_j$) for some coefficients $\sum_{j'} c_{j'}^2 = 1$. Apply first order Taylor expansion of eigenvector definition we have:

$$
\hat{\lambda}_j \hat{U}_j = \hat{M} \hat{U}_j = (M + (MH + G)) \sum_{j'} c_{j'} U_{j'} = \sum_{j'} \lambda_{j'} c_{j'} U_{j'} + (MH + G) \hat{U}_j.
$$

Since we also have $\hat{\lambda}_j \hat{U}_j = \sum_{j'} \hat{\lambda}_j c_{j'} U_{j'}$, we can write $\sum_{j'} (\hat{\lambda}_j - \lambda_{j'}) c_{j'} U_{j'} = (MH + G) \hat{U}_j$, and we can solve for the coefficients $c_{j'}$’s from the linear system as $[(\hat{\lambda}_j - \lambda_{j'}) c_{j'} : j' \in [k]] = U^{-1}(MH + G) \hat{U}_j$. Finally plug in the inequality in (14) we have that for any $j$:

$$
\|\hat{U}_j - U_j\|_2^2 = \sum_{j' \neq j} c_{j'}^2 \|U_{j'}\|_2^2 + (c_j - 1)^2 \|U_j\|_2^2
$$

$$
\leq 2 \sum_{j' \neq j} c_{j'}^2 \|V_{j'}\|_2^2
$$

$$
\leq 8 \|U^{-1}(MH + G)\|_2^2 \frac{\|\hat{U}_j\|_2^2}{\text{sep}(D)^2} \|U_j\|_2 \|V_j\|_2^2
$$

$$
\leq 8 (\sigma_{\max}(D) \sigma_{\max}(H) + \sigma_{\max}(G))^2 \|\hat{U}_j\|_2^2 \|V_j\|_2^2
$$

$$
\leq 8 \frac{\sigma_{\max}(D) \sigma_{\max}(H) + \sigma_{\max}(G)^2}{\sigma_{\min}(U)^2 \text{sep}(D)^2} \|\hat{U}_j\|_2^2 \|V_j\|_2^2.
$$

3. Note that in the above bound for $\|\hat{U}_j - U_j\|$, we can bound the perturbation matrices $H$ and $G$ by:

$$
\sigma_{\max}(H) \leq \frac{\|Z_2\|}{(1 - \sigma_{\max}(E_2^{-1} Z_2)) \sigma_{\min}(E_2)} \leq \frac{\|Z_2\|}{\sigma_{\min}(E_2)^2} \|Z_2\| \leq \frac{\|Z_2\|}{\sigma_{\min}(E_2)^2 \sigma_{\min}(D_2) - \|Z_2\|},
$$

$$
\sigma_{\max}(G) \leq \frac{\sigma_{\max}(Z_1)}{\sigma_{\min}(E_2)} \leq \frac{\|Z_2\|}{\sigma_{\min}(E_2)^2 \sigma_{\min}(D_2)},
$$

Note that $\sigma_{\min}(D_2) \geq w_{\min}$ and $\sigma_{\max}(D) = 1$ by definition. In the following claim, we apply anti-concentration bound to show that with high probability $\text{sep}(D)$ is large.

Claim 3.7. For any $\delta_v \in (0, 1)$, with probability at least $1 - \delta_v$, we can bound $\text{sep}(D)$ by:

$$
\text{sep}(D) \geq \frac{\Delta \delta_v}{\sqrt{d k^2}}.
$$
Proof. Denote \( v = v^{(1)} - v^{(2)} \), and note that \( \|v\| \leq \sqrt{2} \). In the regime we concern, for any pair \( j \neq j' \), we have \( |e^{i\pi < \mu(j), v>} - e^{i\pi < \mu(j'), v>}| \leq |< \mu(j) - \mu(j'), v >| \). Apply Lemma 4.3, we have that for \( \delta \in (0, 1) \),

\[
P( \| \mu(j) - \mu(j') \| \leq \|v\| \leq \sqrt{2} ) \leq \delta.
\]

Take a union bound over all pairs of \( j \neq j' \), we have that

\[
P \left( \text{for some } j \neq j', \| v \| \leq \| \mu(j) - \mu(j') \| \leq \frac{\delta}{\sqrt{d}} \right) \leq \delta.
\]

Recall that \( \Delta = \min_{j \neq j'} \| \mu(j) - \mu(j') \| \).

4. Recall that \( U = P^\top V \). Note that since \( P \) has orthonormal columns, we have \( \sigma_{\min}(U) = \sigma_{\min}(V) \) and \( \| U_i \| \leq \| V_i \| = \sqrt{m} \).

Finally we apply perturbation bound to the estimates \( \hat{V}_i = \hat{P} U_i \) and conclude with the above inequalities:

\[
\| \hat{V}_i - V_i \| \leq 2 \left( \| \hat{P} - P \| \| U_i \| + \| P \| \| \hat{U}_i - U_i \| \right)
\leq 2 \left( \frac{\epsilon \sqrt{m}}{w_{\min} \sigma_{\min}(V)^2} + 3 \frac{\sigma_{\max}(H) \sigma_{\max}(D) + \sigma_{\max}(G)}{\sigma_{\min}(U)^2} \| V_i \| \right) \| V_i \|
\leq 2 \left( \frac{\epsilon \sqrt{m}}{w_{\min} \sigma_{\min}(V)^2} + 6 \frac{\sigma_{\min}(V)^2 \sigma_{\min}(D) - \| Z \| \sigma_{\min}(V)^2 \sigma_{\min}(D)^2}{\sigma_{\min}(V)^2 \sigma_{\min}(D)} \| V_i \| \right) \| V_i \|
\leq C \frac{\sqrt{dk^2 m \ w_{\max} \sigma_{\min}(V)^2}}{\Delta \epsilon} \| V_i \| \epsilon,
\]

for some universal constant \( C \). Note that the last inequality used the assumption that \( \epsilon \) is small enough.

**Condition number of \( V_S' \):** The following lemma is helpful:

**Lemma 3.8.** Let \( V_S' \in \mathbb{C}^{(m+d+1) \times k} \) be the factor as defined in (11). Recall that \( V_S' = [V_S; V_d; 1] \), where \( V_d \) is defined in (8), and \( V_S \) is the characteristic matrix defined in (10).

We can bound the condition number of \( V_S' \) by

\[
\text{cond}_2(V_S') \leq \sqrt{1 + k \text{cond}_2(V_S)}.
\]

**Proof.** (of Lemma 3.8) By definition, there exist some constants \( \lambda \) and \( \lambda' \) such that \( \text{cond}_2(V_S) = \lambda' / \lambda \), and for all \( w \in \mathcal{P}_{1,2}^d \), we have \( \lambda \leq \| V_S w \| \leq \lambda' \). Note that each element of the factor \( V_S' \) lies on the unit circle in the complex plane, then we have:

\[
\lambda^2 \leq \| V_S w \|^2 \leq \| V_S' w \|^2 \leq (\lambda')^2 + \sqrt{kd}.
\]

We can bound the condition number of \( V_S' \) by:

\[
\text{cond}_2(V_S') \leq \sqrt{(\lambda')^2 + \sqrt{kd} / \lambda^2} \leq \sqrt{1 + \sqrt{kd} \text{cond}_2(V_S)} \leq \sqrt{1 + \sqrt{kd} \text{cond}_2(V_S)},
\]

where the last inequality is because that \( \max_w \| V_S w \|^2 \geq \| V_S w \|^2 \geq \| V_S e_1 \|^2 = d \), we have \( (\lambda')^2 \geq d \).

\[\Box\]
**Condition number of the characteristic matrix $V_S$:** Therefore, the stability analysis of the proposed algorithm boils down to understanding the relation between the random sampling set $S$ and the condition number of the characteristic matrix $V_S$. This is analyzed in Lemma 3.10 (main technical lemma).

**Lemma 3.9.** For any fixed number $\epsilon_x \in (0, 1/2)$. Consider a Gaussian vector $s$ with distribution $\mathcal{N}(0, R^2I_{d\times d})$, where $R \geq \sqrt{\frac{2\log(k/\epsilon_x)}{\pi\Delta}}$ for $d \geq 2$, and $R \geq \sqrt{\frac{2\log(1+2/\epsilon_x)}{\pi\Delta}}$ for $d = 1$. Define the Hermitian random matrix $X_s \in \mathbb{C}_{d\times d}^{\text{herm}}$ to be

$$X_s = \begin{bmatrix} e^{-i\pi<\mu(1),s>} & e^{-i\pi<\mu(2),s>} & \cdots & e^{-i\pi<\mu(k),s>} \\ e^{-i\pi<\mu(2),s>} & \ddots & \vdots & \vdots \\ \vdots & \ddots & e^{-i\pi<\mu(k),s>} & \vdots \\ e^{-i\pi<\mu(k),s>} & \cdots & e^{-i\pi<\mu(k),s>} & e^{-i\pi<\mu(1),s>} \end{bmatrix}.$$  \hspace{1cm} (16)

We can bound the spectrum of $E_s[X_s]$ by:

$$1 - \epsilon_x)I_{k\times k} \preceq E_s[X_s] \preceq (1 + \epsilon_x)I_{k\times k}. \hspace{1cm} (17)$$

**Proof.** (of Lemma 3.9) Denote $Y = E_s[X_s]$. Note that $Y_{j,j} = 1$ for all diagonal entries. For $d = 1$ case, the point sources all lie on the interval $[-1, 1]$, we can bound the summation of the off diagonal entries in the matrix $Y$ by:

$$\sum_{j' \neq j} |Y_{j,j'}| = E_s[e^{i\pi<\mu(j')-\mu(j),s>}]$$

$$= \sum_{j' \neq j} e^{-\frac{1}{2}\pi^2\|\mu(j')-\mu(j)\|_2^2 R^2}$$

$$\leq 2(e^{-\frac{1}{2}(\pi\Delta R)^2} + e^{-\frac{1}{2}(\pi(2\Delta)R)^2} + \cdots + e^{-\frac{1}{2}(\pi(k/2)\Delta R)^2})$$

$$\leq 2e^{-\frac{1}{2}(\pi\Delta R)^2}/(1 - e^{-\frac{1}{2}(\pi\Delta R)^2})$$

$$\leq \epsilon_x.$$  \hspace{1cm} (18)

For $d \geq 2$ case, we simply bound each off-diagonal entries by:

$$Y_{j,j'} = e^{-\frac{1}{2}\pi^2\|\mu(j)-\mu(j')\|_2^2 R^2} \leq e^{-\frac{1}{2}\pi^2\Delta^2 R^2} \leq \epsilon_x/k.$$  \hspace{1cm} (18)

Apply Lemma 4.2 (Gershgorin’s Disk Theorem) and we know that all the eigenvalues of $Y$ are bounded by $1 \pm \epsilon_x$. \hfill $\square$

**Lemma 3.10 (Main technical lemma).** In the same setting of Lemma 3.9. Let $S = \{s^{(1)}, \ldots, s^{(m)}\}$ be $m$ independent samples of the Gaussian vector $s$. For $m \geq \frac{k}{\epsilon_x} \sqrt{8 \log \frac{k}{\delta_x}}$, with probability at least $1 - \delta_x$ over the random sampling, the condition number of the factor $V_S$ is bounded by:

$$\text{cond}_2(V_S) \leq \sqrt{\frac{1+2\epsilon_x}{1-2\epsilon_x}}. \hspace{1cm} (18)$$

**Proof.** (of Lemma 3.10) Let $\{X^{(1)}, \ldots, X^{(m)}\}$ denote the i.i.d. samples of the random matrix $X_s$ defined in (16), with $s$ evaluated at the i.i.d. random samples in $S$. Note that we have

$$\|V_Sw\|_2^2 = w^*V_S^*V_Sw = w^*\left(\frac{1}{m}\sum_{i=1}^{m} X^{(i)}\right)w.$$  \hspace{1cm} (18)
By definition of condition number, to show that \( \text{cond}_2(V_S) \leq \sqrt{1 + 2\epsilon_x}, \) it suffices to show that

\[
(1 - 2\epsilon_x)I_{k \times k} \preceq \left( \frac{1}{m} \sum_{i=1}^{m} X^{(i)} \right) \preceq (1 + 2\epsilon_x)I_{k \times k}.
\]

By Lemma 3.9, the spectrum of \( E_s[X_s] \) lies in \( (1 - \epsilon_x, 1 + \epsilon_x) \). Here we only need to show that the spectrum of the sample mean \( \left( \frac{1}{m} \sum_{i=1}^{m} X^{(i)} \right) \) is close to the spectrum of the expectation \( E_s[X_s] \). Since each element of the random matrix \( X_s \in \mathbb{C}^{k \times k} \) lies on the unit circle in the complex plane, we have \( X^2_s \preceq k^2I \) almost surely. Therefore we can apply Lemma 4.1 (Matrix Hoeffding) to show that for \( m > \frac{k}{\epsilon_x} \sqrt{8 \log \frac{k}{\delta_s}} \), with probability at least \( 1 - \delta_s \), it holds that \( \| \frac{1}{m} \sum_{i=1}^{m} X^{(i)} - E_s[X_s] \|_2 \leq \epsilon_x \).

### 4 Discussions

#### 4.1 Numerical results

We empirically demonstrate the performance of the proposed super-resolution algorithm in this section.

First, we look at a simple instance with dimension \( d = 2 \) and the minimal separation \( \Delta = 0.05 \). Our perturbation analysis of the stability result limits to small noise, i.e. \( \epsilon_x \) is inverse polynomially small in the dimensions, and the number of measurements \( m \) needs to be polynomially large in the dimensions. However, we believe these are only the artifact of the crude analysis, instead of being intrinsic to the approach. In the following numerical example, we examine a typical instance of 8 randomly generated 2-D point sources. The minimal separation \( \Delta \) is set to be 0.01, and the weights are uniformly distributed in \([0.1, 1.1]\). The measurement noise level \( \epsilon_z \) is set to be 0.1, and we take only 2178 noisy measurements \( (\ll 1/\Delta^2) \). Figure 1 shows reasonably good recovery result.

![Figure 1](image.png)

Figure 1: The xy plane shows the coordinates of the point sources: true point sources (cyan), the two closest points (blue), and the estimated points (red); the z axis shows the corresponding mixing weights. Dimension \( d = 2 \), number of point sources \( k = 8 \), minimal separation \( \Delta = 0.05 \) and the measurement noise level \( \epsilon_z = 0.1 \); we set the cutoff frequency \( R = 200 \) (in the same order as \( 1/\Delta \)), take 2178 random measurements \( (\ll 1/\Delta^2) \).

Next, we examine the phase transition properties implied by the main theorem.
Figure 2 shows the dependency between the cutoff frequency and the minimal separation. For each fixed pair of the minimal separation and the cutoff frequency \((\Delta, R)\), we randomly generate \(k = 8\) point sources in 4-dimensional space while maintaining the same minimal separation. The weights are uniformly distributed in \([0.1, 1.1]\). The recovery is considered successful if the error \(\sum_{j \in [k]} \sqrt{\|\hat{\mu}^{(j)} - \mu^{(j)}\|_2^2} \leq 0.1\) (on average it tolerates around 4\% error per coordinate per point source). This process is repeated 50 times and the rate of success was recorded. Figure 2 plots the success rate in gray-scale, where 0 is black and 1 is white.

We observe that there is a sharp phase transition characterized by a linear relation between the cutoff frequency and the inverse of minimal separation, which is implied by Theorem 3.2.

4.2 Connection with learning GMMs

One reason we are interested in the scaling of the algorithm with respect to the dimension \(d\) is that it naturally leads to an algorithm for learning Gaussian mixture models (GMMs).

In a similar setup, we examine the success rate while varying the minimal separation \(\Delta\) and the number of measurement \(m\).

In Figure 3 we observe that there is a threshold of \(m\) below which the number of measurements is too small to achieve stable recovery; when \(m\) is above the threshold, the success rate increases with the number of measurements as the algorithm becomes more stable. However, note that given the appropriately chosen cutoff frequency \(R\), the number of measurements required does not depend on the minimal separation, and thus the computation complexity does not depend on the minimal separation either.

![Figure 2: Fix dimension \(d = 4\), number of point sources \(k = 8\), number of measurements \(m = k^2\), and the measurement noise level \(\epsilon_z = 0.02\). We vary the minimal separation such that \(\Delta\) ranges from 0.005 to 0.1, and we vary the cutoff frequency \(R\) from 0 to 25. For each pair of \((\frac{1}{\Delta}, R)\) we randomly generate \(k\) point sources and run the proposed algorithm to recover the point sources. The recovery is considered successful if the error \(\sum_{j \in [k]} \sqrt{\|\hat{\mu}^{(j)} - \mu^{(j)}\|_2^2} \leq 0.1\). This process is repeated 50 times and the rate of success was recorded.](image)
Figure 3: Fix dimension $d = 4$, number of point sources $k = 8$, and the measurement noise level $\varepsilon_z = 0.03$. We vary the minimal separation such that $\Delta$ ranges from 0.01 to 0.2, and we use the corresponding cutoff frequency $R = \frac{0.26}{\Delta}$. We also vary the number of measurements $m$ from 4 to 64. For each pair of $(\Delta, m)$ we randomly generate $k$ point sources and run the proposed algorithm to recover the point sources. The recovery is considered successful if the error

$$\sum_{j \in [k]} \left\| \hat{\mu}^{(j)} - \mu^{(j)} \right\|_2^2 \leq 0.1.$$  

This process is repeated 50 times and the rate of success was recorded.

means and the covariance matrices of these Gaussian components, as well as the mixing weights. We denote the parameters by $\{(w_j, \mu^{(j)}, \Sigma^{(j)}) \}_{i \in [k]}$ where the mean vectors $\mu^{(j)} \in [-1, +1]^d$, the covariance matrices $\Sigma^{(j)} \in \mathbb{R}^{d \times d}$ and the mixing weights $w_j \in \mathbb{R}_+$. Learning mixture of Gaussians is a fundamental problem in statistics and machine learning, whose study dates back to Pearson\cite{pearson1899} in the 1900s, and later arise in numerous areas of applications.

In this brief discussion, we only consider the case where the components are spherical Gaussians with common covariance matrices, namely $\Sigma^{(j)} = \sigma^2 I_{d \times d}$ for all $j$. Moreover, we define the separation $\Delta_G$ by:

$$\Delta_G = \min_{j \neq j'} \frac{\left\| \mu^{(j)} - \mu^{(j')} \right\|_2}{\sigma},$$

and we will focus on the well-separated case where $\Delta_G$ is sufficiently large. This class of well-separated GMMs is often used in data clustering.

By the law of large numbers, for large $d$, the probability mass of a $d$-dimensional Gaussian distribution tightly concentrates within a thin shell with a $\sqrt{d} \sigma$ distance from the mean vector. This concentration of distance leads to a line of works of provably learning GMMs in the well-separated case, started by the seminal work of Dasgupta\cite{dasgupta2003} (spherical and identical $\Sigma$, $\Delta_G \geq \Omega(\sqrt{d})$, complexity $\text{poly}(d, k)$) and followed by works of Dasgupta & Schulman \cite{dasgupta2011} (spherical and identical $\Sigma$, $d \gg \log(k)$, $\Delta_G \geq \Omega(d^{1/4})$, complexity $\text{poly}(d, k)$), Arora & Kannan \cite{arora2014} (general and identical $\Sigma$, $\Delta_G \geq \Omega(d^{1/4})$ complexity $O(k^d)$).

Instead of relying on the concentration of distance and use distance based clustering to learn the GMM, we observe that in the well-separated case the characteristic function of the GMM has nice properties, and one can exploit the concentration of the characteristic function to learn the parameters. Note that we do not impose any other assumption on the dimensions $k$ and $d$.  

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Next, we sketch the basic idea of applying the proposed super-resolution algorithm to learn well-separated GMMs, guaranteeing that $N$ the required number of samples from the GMM, as well as the computation complexity both are in the order of $\text{poly}(d, k)$. Since $\sigma$ is a bounded scalar parameter, we can simply apply grid-search to find the best match. In the following we assume that the $\sigma$ is given and focus on learning the mean vectors and the mixing weights.

Evaluate the characteristic function of a $d$ dimensional Gaussian mixture $X$, with identical and spherical covariance matrix $\Sigma = \sigma^2 I_{d \times d}$, at $s \in \mathbb{R}^d$:

$$\phi_X(s) = \mathbb{E}[e^{i<s,x>}] = \sum_{j \in [k]} w_j e^{-\frac{1}{2} \sigma^2 \|s\|^2 + i\mu(j),s>}. $$

Also we let $\hat{\phi}_X(s)$ denote the empirical characteristic function evaluated at $s$ based on $N$ i.i.d. samples $\{x_1, \ldots, x_N\}$ drawn from this GMM:

$$\hat{\phi}_X(s) = \frac{1}{N} \sum_{l \in [N]} e^{i<x_l,s>}. $$

Note that $|e^{i<x_l,s>}| = 1$ for all samples, thus we can apply Bernstein concentration inequality to the characteristic function and argue that $|\hat{\phi}_X(s) - \phi_X(s)| \leq O\left(\frac{1}{\sqrt{N}}\right)$ for all $s$.

In order to apply the proposed super-resolution algorithm, define

$$f(s) = e^{\frac{1}{2} \sigma^2 \pi^2 \|s\|^2} \hat{\phi}_X(\pi s) = \sum_{j \in [k]} w_j e^{i\mu(j),s>}, \quad \text{and} \quad \tilde{f}(s) = e^{\frac{1}{2} \pi^2 \sigma^2 \|s\|^2} \hat{\phi}_X(s). $$

In the context of learning GMM, taking measurements of $\tilde{f}(s)$ corresponding to evaluating the empirical characteristic function at different $s$, for $\|s\|_{\infty} \leq R$, where $R$ is the cutoff frequency. Note that this implies $\|s\|^2 \leq dR^2$. Therefore, we have that with high probability the noise level $\epsilon_z$ can be bounded by

$$\epsilon_z = \max_{\|s\|_{\infty} \leq R} |f(s) - \tilde{f}(s)| = O\left(\frac{e^{\sigma^2 dR^2}}{\sqrt{N}}\right). $$

In order to achieve stable recovery of the mean vector $\mu(j)'s$ using the proposed algorithm, on one hand, we need the cutoff frequency $R = \Omega\left(\frac{1}{\sigma \Delta_G}\right)$; on the other hand, we need the noise level $\epsilon_z = o(1)$, namely having large enough separation $\Delta_G \geq \Omega(d^{1/2})$.

In summary, when the separation condition is satisfied, to achieve target accuracy in estimating the parameters, we need the noise level $\epsilon_z$ to be upper bounded by some inverse polynomial in the dimensions, and this is equivalent to requiring the number of samples from the GMM to be lower bounded by $\text{poly}(k, d)$.

Although this algorithm does not outperform the scaling result in Dasgupta [6], it still sheds light on a different approach of learning GMMs. We leave it as future work to apply super-resolution algorithms to learn more general cases of GMMs or even learning mixtures of log-concave densities.

4.3 Open problems

In a recent work, Chen & Chi [5] showed that via structured matrix completion, the sample complexity for stable recovery can be reduced to $O(k \log^n d)$. However, the computation complexity is still in the order of $O(kd)$ as the Hankel matrix is of dimension $O(k^d)$ and a semidefinite program is used to complete the matrix. It remains an open problem to reduce the sample complexity of our
algorithm from $O(k^2)$ to the information theoretical bound $O(k)$, while retaining the polynomial scaling of the computation complexity.

Recently, Schiebinger et al. [22] studied the problem of learning a mixture of shifted and re-scaled point spread functions $f(s) = \sum_j w_j \varphi(s, \mu_j)$. This model has the Gaussian mixture as a special case, with the point spread function being Gaussian point spread $\varphi(s, \mu_j) = e^{-(s-\mu_j)^\top \Sigma_j^{-1}(s-\mu_j)}$. We have discussed the connection between super-resolution and learning GMM. Another interesting open problem is to generalize the proposed algorithm to learn mixture of broader classes of nonlinear functions.
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References


Auxiliary lemmas

**Lemma 4.1 (Matrix Hoeffding).** Consider a set \( \{X^{(1)}, \ldots, X^{(m)}\} \) of independent, random, Hermitian matrices of dimension \( k \times k \), with identical distribution \( X \). Assume that \( \mathbb{E}[X] \) is finite, and \( X^2 \preceq \sigma^2 I \) for some positive constant \( \sigma \) almost surely, then, for all \( \epsilon \geq 0 \),

\[
\Pr\left( \left\| \frac{1}{m} \sum_{i=1}^{m} X^{(i)} - \mathbb{E}[X] \right\|_2 \geq \epsilon \right) \leq ke^{-\frac{m\epsilon^2}{8\sigma^2}}.
\]

**Proof.** This follows the argument of Lemma 2.2 from Dasgupta & Gupta [7]. Extension to complex number is straightforward as we can bound the real part and the imaginary part separately. \( \square \)

**Lemma 4.2 (Gershgorin’s Disk Theorem).** The eigenvalues of a matrix \( Y \in \mathbb{C}^{k \times k} \) are all contained in the following union of disks in the complex plane: \( \cup_{j=1}^{k} D(Y_{j,j}, R_j) \), where disk \( D(a,b) = \{x \in \mathbb{C}^k : \|x - a\| \leq b\} \) and \( R_j = \sum_{j' \neq j} |Y_{j,j'}| \).

**Lemma 4.3 (Vector Random Projection).** Let \( a \in \mathbb{R}^m \) be a random vector distributed uniformly over \( \mathcal{P}_{1/2}^m \), and fix a vector \( v \in \mathbb{C}^m \). For \( \delta \in (0, 1) \), we have:

\[
\Pr\left( |<a, v>| \leq \frac{\|v\|_2}{\sqrt{em}} \delta \right) \leq \delta
\]

Proof. This follows the argument of Lemma 2.2 from Dasgupta & Gupta [7]. Extension to complex number is straightforward as we can bound the real part and the imaginary part separately. \( \square \)