Learning Exponential Families in High-Dimensions: Strong Convexity and Sparsity

Sham M. Kakade

Department of Statistics The Wharton School University of Pennsylvania, USA

Karthik Sridharan

Toyota Technological Institute Chicago, USA

Ohad Shamir

School of Computer Science and Engineering The Hebrew University of Jerusalem, Israel

Ambuj Tewari

Toyota Technological Institute Chicago, USA

Abstract

The versatility of exponential families, along with their attendant convexity properties, make them a popular and effective statistical model. A central issue is learning these models in high-dimensions, such as when there is some sparsity pattern of the optimal parameter. This work characterizes a certain strong convexity property of *general* exponential families, which allow their generalization ability to be quantified. In particular, we show how this property can be used to analyze generic exponential families under L_1 regularization.

1 Introduction

Exponential models are perhaps the most versatile and pragmatic statistical model for a variety of reasons — modelling flexibility (encompassing discrete variables, continuous variables, covariance matrices, time series, graphical models, etc); convexity properties allowing ease of optimization; and robust generalization ability. A principal issue for applicability to large scale problems is estimating these models when the ambient dimension of the parameters, p, is much larger than the sample size n — the " $p \gg n$ " regime.

Much recent work has focused on this problem in the special case of linear regression in high dimensions, where it is assumed that the optimal parameter vector is sparse (e.g. Zhao and Yu [2006], Candes and Tao [2007], Meinshausen and Yu [2009], Bickel et al. [2008]). This body of prior work focused on: sharply characterizing the convergence rates for the prediction loss; consistent model selection; and obtaining sparse models. As we tackle more challenging problems, there is a growing need for model selection in more general exponential families. Recent work here includes learning Gaussian graphs (Ravikumar et al. [2008b]) and Ising models (Ravikumar et al. [2008a]).

Classical results established that consistent estimation in *general* exponential families is possible, in the asymptotic limit where the number of dimensions is held constant (though some work establishes rates under certain conditions as p is allowed to grow slowly with n [Portnoy, 1988, Ghosal, 2000]). However, in modern problems, we typically grow p rapidly with p (so even asymptotically we are often interested in the regime where $p \gg n$, as in the case of sparse estimation). While we have a handle on this question for a variety of special cases, a pressing question here is understanding how fast p can scale as a function of p in *general* exponential families — such an analysis must quantify the relevant aspects of the particular family at hand which govern their convergence rate. This is the focus of this work. We should emphasize that throughout this paper, while we are interested in

modelling with an exponential family, we are agnostic about the true underlying distribution (e.g we do not necessarily assume that the data generating process is from an exponential family).

Our Contributions and Related Work The key issue in analyzing the convergence rates of exponential families in terms of their prediction loss (which we take to be the log loss) is in characterizing the nature in which they are strictly convex — roughly speaking, in the asymptotic regime where we have a large sample size n (with p kept fixed), we have a central limit theorem effect where the log loss of any exponential family approaches the log loss of a Gaussian, with a covariance matrix corresponding to the Fisher information matrix. Our first main contribution is quantifying the rate at which this effect occurs in general exponential families.

In particular, we show that every exponential family satisfies a certain rather natural growth rate condition on their standardized moments and standardized cumulants (recall that the k-th standardized moment is the *unitless* ratio of the k-th central moment to the k-th power of the standard deviation, which for k=3,4 is the skew and kurtosis). This condition is rather mild, where these moments can grow as fast as k!. Interestingly, similar conditions have been well studied for obtaining exponential tail bounds for the convergence of a random variable to its mean [Bernstein, 1946]. We show that this growth rate characterizes the rate at which the prediction loss of the exponential family behaves as a strongly convex loss function. In particular, our analysis draws many parallels to that of the analysis of Newton's method, where there is a "burn in" phase in which a number of iterations must occur until the function behaves as a locally quadratic function — in our statistical setting, we now require a (quantified) "burn in" sample size, where beyond this threshold sample size, the prediction loss inherits the desired strong convexity properties (i.e. it is locally quadratic).

Our second contribution is an analysis of L_1 regularization in generic families, in terms of both prediction loss and the sparsity level of the selected model. Under a particular sparse eigenvalue condition on the design matrix (the Restricted Eigenvalue (RE) condition in Bickel et al. [2008]), we show how L_1 regularization in general exponential families enjoys a convergence rate of $O(\frac{s\log p}{n})$ (where s is the number of relevant features). This RE condition is one of the least stringent conditions which permit this optimal convergence rate for linear regression case (see Bickel et al. [2008]) — stronger mutual incoherence/irrepresentable conditions considered in Zhao and Yu [2006] also provide this rate. We show that an essentially identical convergence rate can be achieved for *general* exponential families — our results are non-asymptotic and precisely relate n and p.

Our final contribution is one of *approximate* sparse model selection, i.e. where our goal is to obtain a sparse model with low prediction loss. A drawback of the RE condition in comparison to the mutual incoherence condition is that the latter permits perfect recovery of the true features (at the price of a more stringent condition). However, for the case of the linear regression, Zhao and Yu [2006], Bickel et al. [2008] show that, under a sparse eigenvalue or RE condition, the L_1 solution is actually sparse itself (with a multiplicative increase in the sparsity level, that depends on a certain condition number of the design matrix) – so while the the L_1 solution may not precisely recover the true model, it still is sparse (with some multiplicative increase) and does recover those features with large true weights.

For general exponential families, while we do not have a characterization of the sparsity level of the L_1 -regularized solution (an interesting open question), we do however provide a simple two stage procedure (thresholding and refitting) which provides a sparse model, with support on no more than merely 2s features and which has nearly as good performance (with a rather mild increase in the risk) — this result is novel even for the square loss case. Hence, even under the rather mild RE condition, we can obtain both a favorable convergence rate and a sparse model for generic families.

2 The Setting

Our samples $t \in \mathbb{R}^p$ are distributed independently according to D, and we model the process with $P(t|\theta)$, where $\theta \in \Theta$. However, we do not necessarily assume that D lies in this model class. The class of interest is *exponential families*, which, in their natural form, we denote by:

$$P(t|\theta) = h_t \exp\{\langle \theta, t \rangle - \log Z(\theta)\}\$$

where t is the natural sufficient statistic for θ , and $Z(\theta)$ is the partition function. Here, Θ is the natural parameter space — the (convex) set where $Z(\cdot)$ is finite. While we work with an exponential

family in this general (though natural) form, it should be kept in mind that t can be the sufficient statistic for some prediction variable y of interest, or, for a generalized linear model (such as for logistic or linear regression), we can have t be a function of both y and some covariate x (see Dobson [1990]). We return to this point later.

Our prediction loss is the likelihood function and θ^* is the optimal parameter, i.e.

$$\mathcal{L}(\theta) = \mathbb{E}_{t \sim D}[-\log P(t|\theta)], \qquad \theta^* = \operatorname{argmin} \mathcal{L}(\theta).$$

where the argmin is over the natural parameter space and it is assumed that this θ^* is an interior point of this space. Later we consider the case where θ^* is sparse.

We denote the Fisher information of $P(\cdot|\theta^*)$ as $\mathcal{F}^* = \mathbb{E}_{t \sim P(\cdot|\theta^*)} \left[-\nabla^2 \log P(t|\theta^*) \right]$, under the model of θ^* . The induced "Fisher risk" is

$$\|\theta - \theta^{\star}\|_{\mathcal{F}^{\star}}^2 := (\theta - \theta^{\star})^{\top} \mathcal{F}^{\star} (\theta - \theta^{\star}).$$

We also consider the L_1 risk $\|\theta - \theta^*\|_1$.

For a sufficiently large sample size, we expect that the Fisher risk of an empirical minimizer $\hat{\theta}$, $\|\hat{\theta} - \theta^\star\|_{\mathcal{F}^\star}^2$, be close to $\mathcal{L}(\hat{\theta}) - \mathcal{L}(\theta^\star)$ — one of our main contributions is quantifying when this occurs in general exponential families. This characterization is then used to quantify the convergence rate for L_1 methods in these families. We also expect this strong convexity property to be useful for characterizing the performance of other regularization methods as well.

All proofs can be found in the appendix.

3 (Almost) Strong Convexity of Exponential Families

We first consider a certain bounded growth rate condition for standardized moments and standardized cumulants, satisfied by all exponential families. This growth rate is fundamental in establishing how fast the prediction loss behaves as a quadratic function. Interestingly, this growth rate is analogous to those conditions used for obtaining exponential tail bounds for arbitrary random variables.

3.1 Analytic Standardized Moments and Cumulants

Moments: For a univariate random variable z distributed by ρ , let us denote its k-th central moment (centered at the mean) by:

$$m_{k,\rho}(z) = \mathbb{E}_{z \sim \rho} \left[z - m_{1,\rho}(z) \right]^k$$

where $m_{1,\rho}(z)$ is the mean $\mathbb{E}_{z\sim\rho}[z]$. Recall that the k-th standardized moment is the ratio of the k-th central moment to the k-th power of the standard deviation, i.e. $\frac{m_{k,\rho}(z)}{m_{2,\rho}(z)^{k/2}}$. This normalization with respect to standard deviation makes the standard moments unitless quantities. For k=3 and k=4, the standardized moments are the skew and kurtosis.

We now define the *analytic* standardized moment for z — we use the term analytic to reflect that if the moment generating function of z is analytic then z has an analytic moment.

Definition 3.1. Let z be a univariate random variable under ρ . Then z has an analytic standardized moment of α if the standardized moments exist and are bounded as follows:

$$\forall k \ge 3, \ \left| \frac{m_{k,\rho}(z)}{m_{2,\rho}(z)^{k/2}} \right| \le \frac{1}{2} k! \ \alpha^{k-2}$$

(where the above is assumed to hold if the denominator is 0). If $t \in \mathbb{R}^p$ is a multivariate random variable distributed according to ρ , we say that t has an analytic standardized moment of α with respect to a subspace $\mathcal{V} \subset \mathbb{R}^p$ (e.g. a set of directions) if the above bound holds for all univariate $z = \langle v, t \rangle$ where $v \in \mathcal{V}$.

¹Recall that a real valued function is analytic on some domain of \mathbb{R}^p if the derivatives of all orders exist, and if for each interior point, the Taylor series converges in some sufficiently small neighborhood of that point.

This condition is rather mild in that the standardized moments increase as fast as $k!\alpha^{k-2}$ (in a sense α is just a unitless scale, and it is predominantly the k! which makes the condition rather mild). This condition is closely related to those used in obtaining sharp exponential type tail bounds for the convergence of a random variable to its mean — in particular, the Bernstein conditions [Bernstein, 1946] are almost identical to the above, expect that they use the k-th raw moments (not central moments) 2 . In fact, these moment conditions are weaker than requiring "sub-Gaussian" tails.

While we would not expect analytic moments to be finite for all distributions (e.g. heavy tailed ones), we will see that exponential families have (finite) analytic standardized moments.

Cumulants: Recall that the cumulant-generating function f of z under ρ is the log of the moment-generating function, if it exists, i.e. $f(s) = \log \mathbb{E}[e^{sz}]$. The k-th cumulant is given by the k-th derivate of f at 0, i.e. $c_{k,\rho}(z) = f^{(k)}(0)$. The first, second, and third cumulants are just the first, second, and third central moments — higher cumulants are neither moments nor central moments, but rather more complicated polynomial functions of the moments (though these relationships are known). Analogously, the k-th standardized cumulant is $\frac{c_{k,\rho}(z)}{c_{2,\rho}(z)^{k/2}}$ — this normalization with respect to standard deviation (the second cumulant is the variance) makes these unitless quantities.

Cumulants are viewed as equally fundamental as central moments, and we make use of their behavior as well — in certain settings, it is more natural to work with the cumulants. We define the *analytic* standardized cumulant analogous to before:

Definition 3.2. Let z be a univariate random variable under ρ . Then z has an analytic standardized cumulant of α if the standardized cumulants exist and are bounded as follows:

$$\forall k \geq 3, \ \left| \frac{c_{k,\rho}(z)}{c_{2,\rho}(z)^{k/2}} \right| \leq \frac{1}{2} k! \ \alpha^{k-2}$$

(where the above is assumed to hold if the denominator is 0). If $t \in \mathbb{R}^p$ is a multivariate random variable distributed according to ρ , we say that t has an analytic standardized cumulant of α with respect to a subspace $\mathcal{V} \subset \mathbb{R}^p$ if the above bound holds for all univariate $z = \langle v, t \rangle$ where $v \in \mathcal{V}$.

Existence: The following lemma shows that exponential families have (finite) analytic standardized moments and cumulants, as a consequence of the analyticity of the moment and cumulant generating functions (the proof is in the appendix).

Lemma 3.3. If t is the sufficient statistic of an exponential family with parameter θ , where θ is an interior point of the natural parameter space, then t has both a finite analytic standardized moment and a finite analytic standardized cumulant, with respect to all directions in \mathbb{R}^p .

3.2 Examples

Let us consider a few examples. Going through them, there are two issues to bear in mind. First, α is quantified only at a particular θ (later, θ^* is the point we will be interested in) — note that we do not require any uniform conditions on any derivatives over all θ . Second, we are interested in how α could depend on the dimensionality — in some cases, α is dimension free and in other cases (like for generalized linear models), α depends on the dimension through spectral properties of \mathcal{F}^* (and this dimension dependence can be relaxed in the sparse case that we consider, as discussed later).

3.2.1 One Dimensional Families

When θ is a scalar, there is no direction v to consider.

Bernoulli distributions In the canonical form, the Bernoulli distribution is,

$$P(y|\theta) = \exp(y\theta - \log(1 + e^{\theta}))$$

with $\theta \in \mathbb{R} = \Theta$. We have $m_1(\theta^*) = e^{\theta^*}/(1 + e^{\theta^*})$. The central moments satisfy $m_2(\theta^*) = m_1(\theta^*)(1 - m_1(\theta^*))$ and $m_k(\theta^*) \leq m_2(\theta^*)$ for $k \geq 3$. Thus, $\alpha = 1/\sqrt{m_2(\theta^*)}$ is a standardized

²The Bernstein inequalities used in deriving tail bounds require that, for all $k \geq 2$, $\frac{\mathbb{E}[z^k]}{\mathbb{E}[z^2]} \leq \frac{1}{2} k! L^{k-2}$ for some constant L (which has units of z).

analytic moment at any $\theta^* \in \Theta$. Further, $c_k(\theta^*) \leq c_2(\theta^*) = m_2(\theta^*)$ for $k \geq 3$. Thus, α is also a standardized analytic cumulant at any $\theta^* \in \Theta$.

Unit variance Gaussian distributions In the canonical form, unit variance Gaussian is,

$$P(y|\theta) = \exp\left(-\frac{y^2}{2}\right) \exp\left(y\theta - \frac{\theta^2}{2}\right)$$

with $\theta \in \mathbb{R} = \Theta$. We have $m_1(\theta^\star) = \theta^\star$ and $m_2(\theta^\star) = 1$. Odd central moments are 0 and for even $k \geq 4$, we have $m_k(\theta^\star) = \frac{k!}{2^{k/2}(k/2)!}$. Thus, $\alpha = 1$ is a standardized analytic moment at any $\theta^\star \in \Theta$. However, the log-likelihood is already quadratic in this case (as we shall see, there should be no "burn in" phase until it begins to look like a quadratic!). This becomes evident if we consider the cumulants instead. All cumulants $c_k(\theta^\star) = 0$ for $k \geq 3$ and hence $\alpha = 0$ is a standardized analytic cumulant at any $\theta^\star \in \Theta$ — curiously, cumulant generating function cannot be a finite order polynomial of order greater than 2.

3.2.2 Multidimensional Gaussian Covariance Estimation (i.e. "Gaussian Graphs")

Consider a mean zero p-dimensional multivariate Normal parameterized by the precision matrix Θ ,

$$P(Y|\Theta) = \frac{1}{(2\pi)^{p/2}} \exp\left(-\frac{1}{2}\langle\Theta, YY^{\top}\rangle + \log \det(\Theta)\right) .$$

A "direction" here is a positive semi-definite (p.s.d.) matrix V, and we seek the cumulants of the random variable $\langle V, YY^{\top} \rangle$.

Note that YY^{\top} has Wishart distribution $W_p(\Theta^{-1}, 1)$ with the moment generating function,

$$V \mapsto \mathbb{E}\left[\exp\left(\langle V, YY^{\top}\rangle\right)\right] = \det\left(\mathbf{I} - 2V\Theta^{-1}\right)^{-1/2}$$

Let λ_i 's be the eigenvalues of $V\Theta^{-1}$. Then, taking logs, the cumulant generating function f(s),

$$f(s) = \log \mathbb{E}\left[\exp(s\langle V, YY^{\top}\rangle)\right] = \log \prod_{i=1}^{p} (1 - 2s\lambda_i)^{-1/2} = \frac{-1}{2} \sum_{i=1}^{p} \log(1 - 2s\lambda_i)$$
.

The kth derivative of this is

$$f^{(k)}(s) = \frac{1}{2} \sum_{i=1}^{p} \frac{(k-1)!(2\lambda_i)^k}{(1-2s\lambda_i)^k} .$$

Thus, the cumulant $c_{k,\Theta}(V) = f^{(k)}(0) = 2^{k-1}(k-1)! \sum_i \lambda_i^k$. Hence, for $k \geq 3$,

$$\frac{c_{k,\Theta}(V)}{(c_{2,\Theta}(V))^{k/2}} = \frac{2^{k-1}(k-1)! \sum_i \lambda_i^k}{(2 \sum_i \lambda_i^2)^{k/2}} = \frac{1}{2} 2^{k/2} (k-1)! \frac{\sum_i \lambda^{(k/2) \cdot 2}}{(\sum_i \lambda_i^2)^{k/2}} \leq \frac{1}{2} 2^{k/2-1} \cdot k! \; .$$

Thus, $\alpha = \sqrt{2}$ is a standardized analytic cumulant at Θ . Note that it is harder to estimate the central moments in this case. This example is also interesting in connection to the analysis of Newton's method as the function $\log \det(\Theta)$ is self-concordant on the cone of p.s.d. matrices.

3.2.3 Generalized Linear Models

Consider the case where we have some covariate, response pair (X,Y) drawn from some distribution D. Suppose that we have a family of distributions $P(\cdot|\theta;X)$ such that, for each X, it is an exponential family with natural sufficient statistic $t_{y,X}$,

$$P(y|\theta;X) = h_y \exp(\langle \theta, t_{y,X} \rangle - \log Z_X(\theta))$$
,

where $\theta \in \Theta$. The loss we consider is $\mathcal{L}(\theta) = \mathbb{E}_{X,Y \sim D} \left[-\log P(y|\theta;X) \right]$. A special case of this setup is as follows. Say we have a one dimensional exponential family

$$q_{\nu}(y) = h_y \exp(y\nu - \log Z(\nu)) ,$$

where $y, \nu \in R$. The family $P(\cdot | \theta; X)$ can be be simply $q_{(\theta, X)}$ (i.e. taking $\nu = \langle \theta, X \rangle$). Thus,

$$P(y|\theta;X) = h_y \exp(y\langle\theta,X\rangle - \log Z(\langle\theta,X\rangle))$$
.

We see that $t_{y,X} = yX$ and $Z_X(\theta) = Z(\langle \theta, X \rangle)$. For example, when q_ν is either the Bernoulli family or the unit variance Gaussian family, this corresponds to *logistic regression* or *least squares regression*, respectively. It is easy to see that the analogue of having a standardized analytic moment of α at θ w.r.t. a direction v is to have

$$\frac{m_{k,\theta}(v)}{(m_{2,\theta}(v))^{k/2}} \le \frac{1}{2} k! \alpha^{k-2} ,$$

where

$$m_{k,\theta}(v) = \mathbb{E}_X \left[m_{k,P(\cdot|\theta;X)}(\langle t_{y,X}, v \rangle) \right] .$$

In the above equation, the expectation is under $X \sim D_X$, the marginal of D on X. If the sufficient statistic $t_{y,X}$ is bounded by B in the L_2 norm a.s. and the expected Fisher information matrix

$$\mathbb{E}_X \left[\mathbb{E}_{y \sim P(\cdot | \theta; X)} \left[-\nabla^2 \log P(y | \theta; X) \right] \right]$$

has minimum eigenvalue λ_{\min} , then we can choose $\alpha=B/\lambda_{\min}$. Note that λ_{\min} could be small but it arose only because we are considering an arbitrary direction v. If the set of directions $\mathcal V$ is smaller, then we can often get less pessimistic bounds. For example, see section 5.2.2 in the appendix. We also note that similar bounds can be derived when we assume subgaussian tails for $t_{y,X}$ rather than assuming it is bounded a.s.

3.3 Almost Strong Convexity

Recall that a strictly convex function F is strongly convex if the Hessian of F has a (uniformly) lower bounded eigenvalue (see Boyd and Vandenberghe [2004]). Unfortunately, as for all strictly convex functions, exponential families only behave in a strongly convex manner in a (sufficiently small) neighborhood of θ^* . Our first main result quantifies when this behavior is exhibited.

Theorem 3.4. (Almost Strong Convexity) Let α be either the analytic standardized moment or cumulant under θ^* with respect to a subspace \mathcal{V} . For any θ such that $\theta - \theta^* \in \mathcal{V}$, if either

$$\mathcal{L}(\theta) - \mathcal{L}(\theta^*) \le \frac{1}{65\alpha^2} \quad \text{or} \quad \|\theta - \theta^*\|_{\mathcal{F}^*}^2 \le \frac{1}{16\alpha^2}$$

then

$$\frac{1}{4}\|\theta - \theta^\star\|_{\mathcal{F}^\star}^2 \ \leq \ \mathcal{L}(\theta) - \mathcal{L}(\theta^\star) \ \leq \ \frac{3}{4}\|\theta - \theta^\star\|_{\mathcal{F}^\star}^2$$

Suppose θ is an MLE. Both preconditions can be thought of as a "burn in" phase — the idea being that initially a certain number of samples is needed until the loss of θ is somewhat close to the minimal loss; after which point, the quadratic lower bound engages. This is analogous to the analysis of the Newton's method, which quantifies the number of steps needed to enter the quadratically convergent phase (see Boyd and Vandenberghe [2004]). The constants of 1/4 and 3/4 can be made arbitrarily close to 1/2 (with a longer "burn in" phase), as expected under the central limit theorem.

A key idea in the proof is an expansion of the prediction regret in terms of the central moments. We use the shorthand notation of $c_{k,\theta}(\Delta)$ and $m_{k,\theta}(\Delta)$ to denote the cumulants and moments of the random variable $\langle \Delta, t \rangle$ under the distribution $P(\cdot|\theta)$.

Lemma 3.5. (Moment and Cumulant Expansion) Define $\Delta = \theta - \theta^*$. For all $s \in [0, 1]$,

$$\mathcal{L}(\theta^* + s\Delta) - \mathcal{L}(\theta^*) = \sum_{k=2}^{\infty} \frac{1}{k!} c_{k,\theta^*}(\Delta) s^k$$
$$\mathcal{L}(\theta^* + s\Delta) - \mathcal{L}(\theta^*) = \log\left(1 + \sum_{k=2}^{\infty} \frac{1}{k!} m_{k,\theta^*}(\Delta) s^k\right)$$

where the equalities hold if the right hand sides converge.

The proof of this Lemma (in the appendix) is relatively straightforward. The key technical step in the proof of Theorem 3.4 is characterizing when these expansions converge. Note that for $\Delta=\theta-\theta^{\star}$, even if $\|\Delta\|_{\mathcal{F}^{\star}}^2 \leq \frac{1}{16\alpha^2}$ (one of our preconditions), a direct attempt at lower bounding $\mathcal{L}(\theta^{\star}+\Delta)-\mathcal{L}(\theta^{\star})$ using the above expansions with the analytic moment condition would not imply these expansions converge — the proof requires a more delicate argument.

4 Sparsity

We now consider the case where θ^* is sparse, with support S and sparsity level s, i.e.

$$S = \{i : [\theta^*]_i \neq 0\}, \ s = |S|$$

In order to understand when L_1 regularized algorithms (for linear regression) converge at a rate comparable to that of L_0 algorithms (subset selection), Meinshausen and Yu [2009] considered a sparse eigenvalue condition on the design matrix, where the eigenvalues on any small (sparse) subset are bounded away from 0. Bickel et al. [2008] relaxed this condition so that vectors whose support is "mostly" on any small subset are not too small (see Bickel et al. [2008] for a discussion). We also consider this relaxed condition, but now on the Fisher matrix.

Assumption 4.1. (Restricted Fisher Eigenvalues) For a vector δ , let δ_S be the vector such that $\forall i \in S, [\delta_S]_i = \delta_i$ and δ_S is 0 on the other coordinates, and let S^C denote the complement of S. Assume that:

$$\forall \delta \text{ s.t. } \|\delta_{S^C}\|_1 \leq 3\|\delta_S\|_1, \quad \|\delta\|_{\mathcal{F}^*} \geq \kappa_{\min}^* \|\delta_S\|_2 \\ \forall \delta \text{ s.t. } \delta_{S^C} = 0, \quad \|\delta\|_{\mathcal{F}^*} \leq \kappa_{\max}^* \|\delta_S\|_2$$

The constant of 3 is for convenience. Note we only quantify on the support S — a substantially weaker condition than in Meinshausen and Yu [2009], Bickel et al. [2008], which quantify over *all* subsets (in fact, many previous algorithms/analysis actually use this condition on subsets different from S, e.g. Meinshausen and Yu [2009], Candes and Tao [2007], Zhang [2008]).

Furthermore, with regards to our analyticity conditions, our proof shows that the subspace of directions we need to consider is now restricted to the set:

$$\mathcal{V} = \{ v : \| v_{SC} \|_1 \le 3 \| v_S \|_1 \} \tag{1}$$

Under this Restricted Eigenvalue (RE) condition, we can replace the minimal eigenvalue used in Example 3.2.3 by κ_{\min}^{\star} (section 5.2.2 in appendix), which could be significantly smaller.

4.1 Fisher Risk

Consider the following regularized optimization problem:

$$\hat{\theta} = \operatorname{argmin}_{\theta \in \Theta} \widehat{\mathbb{E}}[-\log P(y|\theta)] + \lambda \|\theta\|_1$$
 (2)

where the empirical expectation is with respect to a sample. This reduces to the usual linear regression example (for Gaussian means) and involves the log-determinant in Gaussian graph setting (considered in Ravikumar et al. [2008b]) where θ is the precision matrix (see Example 3.2.2).

Our next main result provides a risk bound, under the RE condition. Typically, the regularization parameter λ is specified as a function of the noise level, under a particular noise model (e.g. for linear regression case, where $Y=\beta X+\eta$ with the noise model $\eta\sim\mathcal{N}(0,\sigma^2)$, λ is specified as $\sigma\sqrt{\frac{\log p}{n}}$ [Meinshausen and Yu, 2009, Bickel et al., 2008]). Here, our theorem is stated in a deterministic manner (i.e. it is a distribution free statement), to explicitly show that an appropriate value of λ is determined by the L_∞ norm of the measurement error, i.e. $\|\mathbb{E}[t]-\widehat{\mathbb{E}}[t]\|_\infty$ — we then easily quantify λ in a corollary under a mild distributional assumption. Also, we must have that this measurement error be (quantifiably) sufficiently small such that our "burn in" condition holds.

Theorem 4.2. (Risk) Suppose that Assumption 4.1 holds and λ satisfies both

$$\|\mathbb{E}[t] - \widehat{\mathbb{E}}[t]\|_{\infty} \le \frac{\lambda}{2} \quad and \quad \lambda \le \frac{1}{100\alpha^{\star 2} \|\theta^{\star}\|_{1}}$$
 (3)

where α^* is the analytic standardized moment or cumulant of θ^* for the subspace \mathcal{V} defined in (1). (Note this setting requires that $\|\mathbb{E}[t] - \widehat{\mathbb{E}}[t]\|_{\infty}$ be sufficiently small). Then if $\hat{\theta}$ is the solution to the optimization problem in (2), the Fisher risk is bounded as follows

$$\frac{1}{4} \|\hat{\theta} - \theta^{\star}\|_{\mathcal{F}^{\star}}^{2} \le \mathcal{L}(\hat{\theta}) - \mathcal{L}(\theta^{\star}) \le \frac{9s\lambda^{2}}{\kappa_{\min}^{\star}}^{2}$$

and the L_1 risk is bounded as follows:

$$\|\hat{\theta} - \theta^*\|_1 \le \frac{24s\lambda}{\kappa_{\min}^*}^2$$

Intuitively, we expect the measurement error $\|\mathbb{E}[t] - \widehat{\mathbb{E}}[t]\|_{\infty}$ to be $O(\sigma\sqrt{\frac{\log p}{n}})$, so we think of $\lambda = O(\sigma\sqrt{\frac{\log p}{n}})$. Note this would recover the usual (optimal) risk bound of $O(\sigma^2\frac{s\log p}{n})$ (i.e. the same rate as an L_0 algorithm, up to the RE constant). Note that the mild dimension dependence enters through the measurement error. Hence, our theorem shows that all exponential families exhibit favorable convergence rates under the RE condition.

The following proposition and corollary quantify this under a mild (and standard) distributional assumption (which can actually be relaxed somewhat).

Proposition 4.3. If t is sub-Gaussian, ie. there exists $\sigma \geq 0$ such that $\forall i$ and $\forall s \in \mathbb{R}$, $\mathbb{E}\left[e^{s(t_i - \mathbb{E}t_i)}\right] \leq e^{\sigma^2 s^2/2}$, then for any $\delta > 0$, with probability at least $1 - \delta$,

$$\|\mathbb{E}[t] - \widehat{\mathbb{E}}[t]\|_{\infty} \le \sigma \sqrt{\frac{\log\left(\frac{p}{\delta}\right)}{n}}$$

Bounded random variables are in fact sub-Gaussian (though unbounded t may also be sub-Gaussian, e.g. Gaussian random variables are obviously sub-Gaussian). The following corollary is immediate.

Corollary 4.4. Suppose the Assumption 4.1 and the sub-Gaussian condition in Proposition 4.3 hold. For any $\delta > 0$, as long as $n \geq K\alpha^{\star 4} \|\theta^{\star}\|_1^2 \sigma^2 \log\left(\frac{p}{\delta}\right)$, (where K is a universal constant), setting $\lambda = 2\sigma\sqrt{\frac{\log\left(\frac{p}{\delta}\right)}{n}}$, we have with probability at least $1 - \delta$,

$$\|\hat{\theta} - \theta^{\star}\|_{\mathcal{F}^{\star}}^{2} \leq \left(\frac{36}{\kappa_{\min}^{\star}^{2}}\right) \frac{\sigma^{2} s \log\left(\frac{p}{\delta}\right)}{n} \quad and \quad \|\hat{\theta} - \theta^{\star}\|_{1} \leq \frac{48\sigma s}{\kappa_{\min}^{\star}^{2}} \sqrt{\frac{\log(\frac{p}{\delta})}{n}}$$

4.2 Approximate Model Selection

An important issue unaddressed by the previous result is the sparsity level of our estimate $\hat{\theta}$. For the linear regression case, Meinshausen and Yu [2009], Bickel et al. [2008] show that the L_1 solution is actually sparse, with a sparsity level of roughly $O((\frac{\kappa_{\max}^*}{\kappa_{\min}^*})^2 s)$, (i.e. the sparsity level increases by a factor which is essentially a condition number squared). In the general setting, we do not have a characterization of the actual sparsity level of the L_1 solution.

However, we now present a two stage procedure, which provides an estimate with support on merely 2s features, with nearly as good risk (Shalev-Shwartz et al. [2009] discuss this issue of trading sparsity for accuracy, but their results are more applicable to settings with $O(\frac{1}{\sqrt{n}})$ rates.). Consider the procedure where we select the set of coordinates which have large weight under $\hat{\theta}$ (say greater than some threshold τ). Then we refit to find an estimate with support only on these coordinates. That is, we restrict our estimate to the set $\Theta_{\tau} = \{\theta \in \Theta: \theta_i = 0 \text{ if } |\hat{\theta}_i| \leq \tau\}$. This algorithm is:

$$\tilde{\theta} = \operatorname{argmin}_{\theta \in \Theta_{\tau}} \hat{\mathcal{L}}(\theta) + \lambda \|\theta\|_{1} \tag{4}$$

Theorem 4.5. (Sparsity) Suppose that 4.1 holds and the regularization parameter λ satisfies both

$$\|\mathbb{E}[t] - \widehat{\mathbb{E}}[t]\|_{\infty} \le \frac{\lambda}{2} \quad and \quad \lambda \le \min\{\frac{1}{270\alpha^{\star 2}\|\theta^{\star}\|_{1}}, \frac{\kappa_{\min}^{\star}^{2}}{340\kappa_{\max}^{\star}\alpha^{\star}\sqrt{s}}\}$$
 (5)

where α^* is the analytic standardized moment or cumulant of θ^* for the subspace \mathcal{V} defined in (1). If $\hat{\theta}$ is the solution of (2) with this λ and $\tilde{\theta}$ is the solution of (4) with threshold $\tau = \frac{18\lambda}{\kappa_{\min}^*}^2$ and this λ , then:

- 1. $\tilde{\theta}$ has support on at most 2s coordinates.
- 2. The Fisher risk is bounded as follows:

$$\frac{1}{4} \|\hat{\theta} - \theta^{\star}\|_{\mathcal{F}^{\star}}^{2} \leq \mathcal{L}(\hat{\theta}) - \mathcal{L}(\theta^{\star}) \leq \left(12 \frac{\kappa_{\max}^{\star}}{\kappa_{\min}^{\star}}\right)^{2} \frac{9 s \lambda^{2}}{\kappa_{\min}^{\star}^{2}}$$

Using Proposition 4.3, we have following corollary.

Corollary 4.6. Suppose the Assumption 4.1 and the sub-Gaussian condition in Proposition 4.3 hold. Then for any $\delta > 0$, as long as $n \ge K\alpha^{\star 2}\sigma^2\log\left(\frac{p}{\delta}\right)\max\left\{\frac{s\kappa_{\max}^{\star}^2}{\kappa_{\min}^{\star}^4},\alpha^{\star 2}\|\theta^{\star}\|_1^2\right\}$ (where K is a universal constant), setting $\lambda = 2\sqrt{\frac{\sigma^2\log\left(\frac{p}{\delta}\right)}{n}}$ and threshold $\tau = 36\sqrt{\frac{\sigma^2\log\left(\frac{p}{\delta}\right)}{n\kappa_{\min}^{\star}^{\star}}}$, we have that with probability at least $1 - \delta$,

$$\|\tilde{\theta} - \theta^{\star}\|_{\mathcal{F}^{\star}}^{2} \le \left(12 \frac{\kappa_{\max}^{\star}}{\kappa_{\min}^{\star}}\right)^{2} \left(\frac{36}{\kappa_{\min}^{\star}}\right) \frac{s\sigma^{2} \log\left(\frac{p}{\delta}\right)}{n}$$

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5 Appendix

5.1 Proofs for Section 3

Proof. (of Lemma 3.3) The proof shows that the central moment generating function of $z = \langle v, t \rangle$, namely $\mathbb{E}[\exp(s(\langle v, t \rangle - \mathbb{E}[\langle v, t \rangle]))]$, is analytic at θ . First, notice that

$$\mathbb{E}[\exp(s(\langle v, t \rangle - \mathbb{E}[\langle v, t \rangle]))] = \exp(-s\mathbb{E}[\langle v, t \rangle]) \int_{t} h_{t} \exp(s\langle v, t \rangle) \exp\{\langle \theta, t \rangle - \log Z(\theta)\} dt$$

$$= \exp(-s\mathbb{E}[\langle v, t \rangle]) \frac{\int_{t} h_{t} \exp\{\langle \theta + sv, t \rangle\} dt}{\int_{t} h_{t} \exp\{\langle \theta, t \rangle\} dt}$$

$$= \exp(-s\mathbb{E}[\langle v, t \rangle]) \frac{Z(\theta + sv)}{Z(\theta)}.$$

It is known that for exponential families, $Z(\theta)$ (namely, the partition function) is analytic in the interior of Θ (see Brown [1986]). Since $\exp(-s\mathbb{E}[\langle v,t\rangle])$ is also analytic (as a function of s), we have by the chain of equalities above that the central moment generating function is also analytic (as a function of s) for any θ at the interior of Θ . This property implies that the derivatives of the central moment generating function at s=0 (namely, the moments $m_{k,\rho}(z)$) cannot grow too fast with k. In particular, by proposition 2.2.10 in Krantz and Parks [2002], it holds for all k that the k-th derivative (which is equal to $m_{k,\rho}(z)$) is at most $k!B^k$ for some constant B. As a result, $|m_{k,\rho}(z)/m_{2,\rho}(z)^{k/2}|$ is at most $\frac{1}{2}k!\alpha^{k-2}$ for a suitable constant α . Thus, t has finite analytic standardized moment with respect to all directions.

As to the assertion about t having finite analytic standardized cumulant, notice that our argument above also implies that the (raw) moment generating function, $\mathbb{E}[\exp(s\langle v,t\rangle)]$, is analytic. Therefore, $\log(\mathbb{E}[\exp(s\langle v,t\rangle)])$, which is the cumulant generating function, is also analytic (since the logarithm is an analytic function). An analysis completely identical to the above leads to the desired conclusion about the cumulants of t.

From here on, we slightly abuse notation and let $m_k(\Delta)$ be the k-th central moment of the univariate random variable $\langle \Delta, t \rangle$ distributed under θ^* .

Proof. (of Lemma 3.5) First, note that since θ^* is optimal, we have $\mathbb{E}_{t \sim D}[t] = \mathbb{E}_{t \sim P(\cdot \mid \theta^*)}[t]$. Hence,

$$\mathcal{L}(\theta^* + s\Delta) - \mathcal{L}(\theta^*) = -s\langle \Delta, \mathbb{E}_{t \sim P(\cdot | \theta^*)}[t] \rangle + \log \frac{Z(\theta^* + s\Delta)}{Z(\theta^*)}$$
$$= -sm_1(\Delta) + \log \frac{Z(\theta^* + s\Delta)}{Z(\theta^*)}$$
$$= \log \frac{e^{-sm_1(\Delta)}Z(\theta^* + s\Delta)}{Z(\theta^*)}$$

In the proof of Lemma 3.3 it was shown that $e^{-sm_1(\Delta)}\frac{Z(\theta^*+s\Delta)}{Z(\theta^*)}$ is the central moment generating function, that it is analytic, and that the expression above is analytic as well. Their Taylor expansions complete the proof.

The following upper and lower bounds are useful in that they guarantee the sum converges for the choice of s specified.

Lemma 5.1. Let α and θ be defined as in Theorem 3.4. Let $\Delta = \theta - \theta^*$ and set $s = \min\{\frac{1}{4\alpha\sqrt{m_2(\Delta)}}, 1\}$. If is α is an analytic moment, then

$$\frac{1}{3} \frac{m_2(\Delta)}{\max\{16\alpha^2 m_2(\Delta), 1\}} \, \leq \, \sum_{k=2}^{\infty} \frac{m_k(\Delta) s^k}{k!} \, \leq \, \frac{2}{3} \frac{m_2(\Delta)}{\max\{16\alpha^2 m_2(\Delta), 1\}}$$

If is α is an analytic cumulant, then

$$\frac{1}{3} \frac{c_2(\Delta)}{\max\{16\alpha^2 c_2(\Delta), 1\}} \le \sum_{k=2}^{\infty} \frac{c_k(\Delta) s^k}{k!} \le \frac{2}{3} \frac{c_2(\Delta)}{\max\{16\alpha^2 c_2(\Delta), 1\}}$$

Proof. We only prove the analytic moment case (the proof for the cumulant case is identical). First let us show that:

$$\frac{s^2 m_2(\Delta)}{2} \left(1 - \sum_{k=1}^{\infty} (s\alpha \sqrt{m_2(\Delta)})^k \right) \le \sum_{k=2}^{\infty} \frac{m_k(\Delta) s^k}{k!} \le \frac{s^2 m_2(\Delta)}{2} \left(1 + \sum_{k=1}^{\infty} (s\alpha \sqrt{m_2(\Delta)})^k \right)$$

We can bound the following sum from k = 3 onwards as:

$$\left| \sum_{k=3}^{\infty} \frac{1}{k!} m_k(\Delta) s^k \right| \le \frac{1}{2} \sum_{k=3}^{\infty} \alpha^{k-2} m_2(\Delta)^{\frac{k}{2}} s^k = \frac{s^2 m_2(\Delta)}{2} \sum_{k=1}^{\infty} (s\alpha \sqrt{m_2(\Delta)})^k$$

which proves the claim.

For our choice of s,

$$\sum_{k=1}^{\infty} (s\alpha\sqrt{m_2(\Delta)})^k = \sum_{k=1}^{\infty} \left(\min\left\{\frac{1}{4}, \alpha\sqrt{m_2(\Delta)}\right\}\right)^k \le \sum_{k=1}^{\infty} \left(\frac{1}{4}\right)^k = \frac{1}{3}$$

Hence, we have:

$$\sum_{k=2}^{\infty} \frac{m_k(\Delta)s^k}{k!} \ge \frac{s^2 m_2(\Delta)}{2} \left(1 - \sum_{k=1}^{\infty} (s\alpha\sqrt{m_2(\Delta)})^k \right)$$
$$\ge \frac{s^2 m_2(\Delta)}{3}$$
$$= \frac{1}{3} \frac{m_2(\Delta)}{\max\{16\alpha^2 m_2(\Delta), 1\}}$$

Analogously, the upper bound can be proved.

The following core lemma leads to the proof of Theorem 3.4.

Lemma 5.2. Let α and θ be defined as in Theorem 3.4. We have that:

$$\frac{1}{4} \frac{\|\theta - \theta^{\star}\|_{\mathcal{F}^{\star}}^{2}}{\max\{16\alpha^{2}\|\theta - \theta^{\star}\|_{\mathcal{F}^{\star}}^{2}, 1\}} \le \mathcal{L}(\theta) - \mathcal{L}(\theta^{\star})$$
(6)

Furthermore, if $\|\theta - \theta^*\|_{\mathcal{F}^*} \leq \frac{1}{16\alpha^2}$,

$$\frac{1}{4}\|\theta - \theta^{\star}\|_{\mathcal{F}^{\star}}^{2} \leq \mathcal{L}(\theta) - \mathcal{L}(\theta^{\star}) \leq \frac{2}{3}\|\theta - \theta^{\star}\|_{\mathcal{F}^{\star}}^{2}$$

Proof. As s is clearly in [0, 1] and by convexity, we have:

$$\mathcal{L}(\theta) - \mathcal{L}(\theta^*) = \mathcal{L}(\theta^* + \Delta) - \mathcal{L}(\theta^*)$$
$$\geq \mathcal{L}(\theta^* + s\Delta) - \mathcal{L}(\theta^*)$$

For the cumulant case, we have that this is lower bounded by $\frac{m_2(\Delta)}{3 \max\{16\alpha^2 m_2(\Delta),1\}}$ using Lemma 5.1 and Lemma 3.5, which proves (6). Now consider the analytic moment case. By, Lemma 3.5, we have

$$\mathcal{L}(\theta) - \mathcal{L}(\theta^*) \ge \log(1 + \frac{m_2(\Delta)}{3 \max\{16\alpha^2 m_2(\Delta), 1\}})$$

Now by Jensen's inequality, we know that the fourth standardized moment (the kurtosis) is greater than one, so $\alpha^2 \geq \frac{1}{12}$ (since $\frac{4!}{2}\alpha^2 \geq 1$). This implies that:

$$\frac{m_2(\Delta)}{3 \max\{16\alpha^2 m_2(\Delta), 1\}} \le \frac{1}{48\alpha^2} \le 1/4$$

since the sum is only larger if we choose any argument in the max. Now for $0 \le x \le 1/4$, we have that $\log(1+x) \ge 1 + x - x^2 \ge 1 + \frac{3}{4}x$. Proceeding,

$$\log(1 + \frac{m_2(\Delta)}{3 \max\{16\alpha^2 m_2(\Delta), 1\}}) \ge \frac{m_2(\Delta)}{4 \max\{16\alpha^2 m_2(\Delta), 1\}}$$

which proves (6) (for the analytic moment case).

For the second claim, the precondition implies that the max, in (6), will be achieved with the argument of 1, which directly implies the lower bound. For the upper bound, we can apply Lemma 5.1 with s=1 (s=1 under our precondition), which implies that $\sum_{k=2}^{\infty} \frac{m_k(\Delta)}{k!}$ is less than $\frac{2}{3}m_2(\Delta)$. The claim follows directly for the cumulant case using Lemma 3.5, with s=1. For the moment case, we use that $\log(1+x) \leq x$.

We are now ready to prove Theorem 3.4.

Proof. (of Theorem 3.4) If $\|\theta - \theta^\star\|_{\mathcal{F}^\star}^2 \leq \frac{1}{16\alpha^2}$, then the previous Lemma implies the claim. Let us assume the condition on the loss, i.e. $\mathcal{L}(\theta) - \mathcal{L}(\theta^\star) \leq \frac{1}{65\alpha^2}$. If $\|\theta - \theta^\star\|_{\mathcal{F}^\star}^2 \leq \frac{1}{16\alpha^2}$, then we are done by the previous argument. So let us assume that $\|\theta - \theta^\star\|_{\mathcal{F}^\star}^2 > \frac{1}{16\alpha^2}$. Hence, $\max\{16\alpha^2m_2(\Delta),1\} = 16\alpha^2m_2(\Delta)$. Using (6), we have that $\frac{1}{64\alpha^2} \leq \mathcal{L}(\theta) - \mathcal{L}(\theta^\star)$, which is a contradiction.

5.2 Proofs for Section 4

5.2.1 Proof of Theorem 4.2

Throughout, let $\widehat{\mathcal{L}}(\theta) = \widehat{\mathbb{E}}[-\log P(y|\theta)]$. Also, let $T = \mathbb{E}[t]$ and $\widehat{T} = \widehat{\mathbb{E}}[t]$.

Lemma 5.3. Suppose that (3) holds (i.e. that $||T - \hat{T}||_{\infty} \le \lambda/2$). Let $\hat{\theta}$ be a solution the optimization problem in (2). For all $\theta \in \Theta$, we have:

$$\mathcal{L}(\hat{\theta}) - \mathcal{L}(\theta) \leq \frac{\lambda}{2} \|\hat{\theta} - \theta\|_1 + \lambda \|\theta\|_1 - \lambda \|\hat{\theta}\|_1$$

$$\leq \frac{3\lambda}{2} \|\theta\|_1$$
(7)

Furthermore, suppose that θ only has support on S, then:

$$\mathcal{L}(\hat{\theta}) - \mathcal{L}(\theta) \le \frac{3\lambda}{2} \|\hat{\theta}_S - \theta\|_1 \tag{8}$$

Proof. Since $\hat{\theta}$ solves (2), we have:

$$-\langle \hat{\theta}, \hat{T} \rangle + \log Z(\hat{\theta}) + \lambda \|\hat{\theta}\|_{1} \le -\langle \theta, \hat{T} \rangle + \log Z(\theta) + \lambda \|\theta\|_{1}$$

Hence,

$$-\langle \hat{\theta}, T \rangle + \log Z(\hat{\theta}) + \lambda \|\hat{\theta}\|_{1} \le \langle \hat{\theta} - \theta, \hat{T} - T \rangle - \langle \theta, T \rangle + \log Z(\theta) + \lambda \|\theta\|_{1}$$

Using this and the condition on λ , we have

$$\mathcal{L}(\hat{\theta}) - \mathcal{L}(\theta) \leq \langle \hat{\theta} - \theta, \hat{T} - T \rangle + \lambda \|\theta\|_{1} - \lambda \|\hat{\theta}\|_{1}$$

$$\leq \|\hat{\theta} - \theta\|_{1} \|\hat{T} - T\|_{\infty} + \lambda \|\theta\|_{1} - \lambda \|\hat{\theta}\|_{1}$$

$$\leq \frac{\lambda}{2} \|\hat{\theta} - \theta\|_{1} + \lambda \|\theta\|_{1} - \lambda \|\hat{\theta}\|_{1}$$

which proves the first inequality. Continuing,

$$\frac{\lambda}{2} \|\hat{\theta} - \theta\|_{1} + \lambda \|\theta\|_{1} - \lambda \|\hat{\theta}\|_{1}
\leq \frac{\lambda}{2} (\|\hat{\theta}\|_{1} + \|\theta\|_{1}) + \lambda \|\theta\|_{1} - \lambda \|\hat{\theta}\|_{1}
\leq \frac{3\lambda}{2} \|\theta\|_{1}$$

which proves the next inequality.

For the final claim, using the sparsity assumption on θ , we have:

$$\mathcal{L}(\hat{\theta}) - \mathcal{L}(\theta) \leq \frac{\lambda}{2} \|\hat{\theta} - \theta\|_{1} + \lambda \|\theta\|_{1} - \lambda \|\hat{\theta}\|_{1}$$

$$= \frac{\lambda}{2} \|\hat{\theta}_{S} - \theta\|_{1} + \frac{\lambda}{2} \|\hat{\theta}_{SC}\|_{1} + \lambda \left(\|\theta\|_{1} - \|\hat{\theta}_{S}\|_{1} \right) - \lambda \|\hat{\theta}_{SC}\|_{1}$$

$$\leq \frac{\lambda}{2} \|\hat{\theta}_{S} - \theta\|_{1} + \lambda \|\hat{\theta}_{SC}\|_{1} + \lambda \|\hat{\theta}_{S} - \theta\|_{1} - \lambda \|\hat{\theta}_{SC}\|_{1}$$

$$= \frac{3\lambda}{2} \|\hat{\theta}_{S} - \theta\|_{1}$$

where the second to last step uses the triangle inequality. This completes the proof. \Box

Lemma 5.4. Suppose that (3) holds. Let $\hat{\theta}$ be a solution the optimization problem in (2). For any $\theta \in \Theta$, which only has support on S and such that $\mathcal{L}(\hat{\theta}) \geq \mathcal{L}(\theta)$, then:

$$\|\hat{\theta}_{S^C}\|_1 \le 3\|\hat{\theta}_S - \theta\|_1 \tag{9}$$

$$\|\hat{\theta} - \theta\|_1 \le 4\|\hat{\theta}_S - \theta\|_1 \tag{10}$$

Proof. By assumption on θ and (7),

$$0 \le \mathcal{L}(\hat{\theta}) - \mathcal{L}(\theta) \le \frac{\lambda}{2} \|\hat{\theta} - \theta\|_1 + \lambda \|\theta\|_1 - \lambda \|\hat{\theta}\|_1$$

Dividing by λ and adding $\frac{1}{2} ||\hat{\theta} - \theta||_1$ to both the left and right sides,

$$\frac{1}{2}\|\hat{\theta} - \theta\|_1 \le \|\hat{\theta} - \theta\|_1 + \|\theta\|_1 - \|\hat{\theta}\|_1$$

For any component $i \notin S$, we have that $|\hat{\theta}_i - \theta_i| + |\theta_i| - |\hat{\theta}_i| = 0$. Hence,

$$\frac{1}{2}\|\hat{\theta} - \theta\|_1 \le \|\hat{\theta}_S - \theta\|_1 + \|\theta\|_1 - \|\hat{\theta}_S\|_1 \le 2\|\hat{\theta}_S - \theta\|_1$$

where the last step uses the triangle inequality $(\|\theta\|_1 - \|\hat{\theta}_S\|_1 \le \|\hat{\theta}_S - \theta\|_1)$. This proves (10). From this,

$$\frac{1}{2}\|\hat{\theta}_S - \theta\|_1 + \frac{1}{2}\|\hat{\theta}_{S^C}\|_1 = \frac{1}{2}\|\hat{\theta} - \theta\|_1 \le 2\|\hat{\theta}_S - \theta\|_1$$

which proves (9), after rearranging.

Now we are ready to prove Theorem 4.2.

Proof. (of Theorem 4.2). First, by (3) and (7) we see that

$$\mathcal{L}(\hat{\theta}) - \mathcal{L}(\theta^{\star}) \le \frac{1}{65\alpha^{\star 2}}$$

(note that $\hat{\theta}$ satisfies the RE precondition, so $\hat{\theta} - \theta^* \in \mathcal{V}$). Hence using Theorem 3.4 we see that

$$\frac{1}{4} \|\hat{\theta} - \theta^{\star}\|_{\mathcal{F}^{\star}}^{2} \le \mathcal{L}(\hat{\theta}) - \mathcal{L}(\theta^{\star})$$

On the other hand observe that:

$$\|\hat{\theta}_S - \theta^*\|_1 \le \sqrt{s} \|\hat{\theta}_S - \theta^*\|_2 \le \frac{\sqrt{s}}{\kappa_{\min}^*} \|\hat{\theta} - \theta^*\|_{\mathcal{F}^*}$$
(11)

where the last step uses the Restricted Eigenvalue Condition, Assumption 4.1. Now using the above with (8) we have that

$$\frac{1}{4} \|\hat{\theta} - \theta^{\star}\|_{\mathcal{F}^{\star}}^{2} \leq \mathcal{L}(\hat{\theta}) - \mathcal{L}(\theta^{\star}) \leq \frac{3\lambda\sqrt{s}}{2\kappa_{\min}^{\star}} \|\hat{\theta} - \theta^{\star}\|_{\mathcal{F}^{\star}}$$

Hence,

$$\|\hat{\theta} - \theta^{\star}\|_{\mathcal{F}^{\star}} \le \frac{6\lambda\sqrt{s}}{\kappa_{\min}^{\star}} \tag{12}$$

and so

$$\frac{1}{4} \|\hat{\theta} - \theta^*\|_{\mathcal{F}^*}^2 \le \mathcal{L}(\hat{\theta}) - \mathcal{L}(\theta^*) \le \frac{9\lambda^2 s}{\kappa_{\min}^*}^2$$

which proves the first claim.

Now to conclude the proof note that by Assumption 4.1

$$\kappa_{\min}^{\star} \|\hat{\theta}_S - \theta^{\star}\|_2 \le \|\hat{\theta} - \theta^{\star}\|_{\mathcal{F}^{\star}} \le \frac{6\lambda\sqrt{s}}{\kappa_{\min}^{\star}}$$

Hence by (10) we see that

$$\|\hat{\theta} - \theta^*\|_1 \le 4\|\hat{\theta}_S - \theta^*\|_1 \le 4\sqrt{s}\|\hat{\theta}_S - \theta^*\|_2 \le \frac{24\lambda s}{\kappa_{\min}^*}^2$$

This concludes the proof.

5.2.2 Analytic Standardized Moment for GLM and Sparsity

In the generalized linear model example in Section 3.2.3, we showed that if the sufficient statistics are bounded by B and if \mathcal{F}^* has minimum eigenvalue λ_{\min} , then we can choose $\alpha = B/\lambda_{\min}$. However, when θ^* is sparse we see that in both Theorems 4.2 and 4.5, we only care about α^* the analytic standardized moment/cumulant of the set \mathcal{V} , specified in (1). Given this, it is clear from the exposition in the generalized linear model example in Section 3.2.3 that α^* can be bounded by B/κ_{\min}^* , since all elements of the set \mathcal{V} satisfy Assumption 4.1.

5.2.3 Proof of Theorem 4.5

Lemma 5.5. (Sparsity or Restricted Set) If the threshold $\tau = \frac{18\lambda}{\kappa_{\min}^*}$, then the size of the support of any $\theta \in \Theta_{\tau}$ is at most 2s

Proof. First notice that on the set S thresholding could potentially leave all the s coordinates. On the other hand notice that if we threshold using τ , then the number of coordinates that remain unclipped in the set S^C is bounded by $\|\hat{\theta}_{S^C}\|_1/\tau$. Hence

$$\left|i:|\hat{\theta}_i|>\tau\right|\leq s+\frac{\|\hat{\theta}_{S^C}\|_1}{\tau}$$

By (9), (12) and the RE assumption, we have

$$\|\hat{\theta}_{S^C}\|_1 \le 3\|\hat{\theta}_S - \theta^*\|_1 \le 3\sqrt{s}\|\hat{\theta}_S - \theta^*\|_2 \le \frac{18\lambda s}{\kappa_{\min}^*}^2$$

Using this we see that

$$\left|i: |\hat{\theta}_i| > \tau \right| \le s + \frac{18\lambda s}{\kappa_{\min}^{\star 2} \tau}$$

Plugging in the value of τ we get the statement of the lemma since support size of $\hat{\theta}^{\tau}$ upper bounds the support size of any $\theta \in \Theta_{\tau}$.

Lemma 5.6. (Bias) Choose $\tau = \frac{18\lambda}{\kappa_{\min}^{*}^{*}}^{2}$. Then,

$$\mathcal{L}(\hat{\theta}_S^{\tau}) - \mathcal{L}(\theta^*) \le \frac{540\kappa_{\max}^* {}^2 s \lambda^2}{\kappa_{\min}^*}$$

where $\hat{\theta}^{\tau}$ is defined as $\hat{\theta}_{i}^{\tau} = \hat{\theta}_{i} \mathbf{1}_{(\hat{\theta}_{i} > \tau)}$.

Proof. Note that

$$\begin{split} \|\hat{\theta}_{S}^{\tau} - \theta^{\star}\|_{\mathcal{F}^{\star}}^{2} &\leq \kappa_{\max}^{\star}{}^{2} \|\hat{\theta}_{S}^{\tau} - \theta^{\star}\|_{2}^{2} \\ &\leq 2\kappa_{\max}^{\star}{}^{2} \left(\|\hat{\theta}_{S}^{\tau} - \hat{\theta}_{S}\|_{2}^{2} + \|\hat{\theta}_{S} - \theta^{\star}\|_{2}^{2} \right) \\ &\leq 2\kappa_{\max}^{\star}{}^{2} \left(s\tau^{2} + \|\hat{\theta}_{S} - \theta^{\star}\|_{2}^{2} \right) \\ &\leq 2\kappa_{\max}^{\star}{}^{2} \left(s\tau^{2} + \frac{36s\lambda^{2}}{\kappa_{\min}^{\star}{}^{4}} \right) \end{split}$$

Where the last step is obtained by applying Theorem 4.2. Substituting for τ ,

$$\|\hat{\theta}_S^{\tau} - \theta^{\star}\|_{\mathcal{F}^{\star}}^2 \le \frac{720\kappa_{\max}^{\star}{}^2 s\lambda^2}{\kappa_{\min}^{\star}}$$
(13)

Now the condition on λ in (5) implies that Theorem 3.4 is applicable, which completes the proof. \Box

Proof of Theorem 4.5. The first claim of the theorem follows from Lemma 5.5. We prove the second claim of the theorem by considering two cases. First, when $\mathcal{L}(\tilde{\theta}) \leq \mathcal{L}(\hat{\theta}_S^{\tau})$. In this case by Lemma 5.6 we have

$$\mathcal{L}(\tilde{\theta}) - \mathcal{L}(\theta^*) \le \frac{540\kappa_{\max}^{*2} s\lambda^2}{\kappa_{\infty}^{*1}}$$

Also by (5), applying Theorem 3.4, we see that

$$\frac{1}{4} \|\tilde{\theta} - \theta^\star\|_{\mathcal{F}^\star}^2 \leq \mathcal{L}(\tilde{\theta}) - \mathcal{L}(\theta^\star) \leq \frac{540 \kappa_{\max}^\star{}^2 s \lambda^2}{\kappa_{\min}^\star{}^4}$$

which gives us the second claim of the theorem. The next case is when $\mathcal{L}(\tilde{\theta}) > \mathcal{L}(\hat{\theta}_S^{\tau})$. In this case, by applying Lemma 5.3 with $\theta = \hat{\theta}_S^{\tau}$, we see that

$$\begin{split} \mathcal{L}(\tilde{\theta}) - \mathcal{L}(\hat{\theta}_S^{\tau}) &\leq \frac{3\lambda}{2} \|\hat{\theta}_S^{\tau}\|_1 \leq \frac{3\lambda}{2} \|\theta^{\star} - \hat{\theta}_S^{\tau}\|_1 + \frac{3\lambda}{2} \|\theta^{\star}\|_1 \\ &\leq \frac{3\lambda\sqrt{s}}{2} \|\theta^{\star} - \hat{\theta}_S^{\tau}\|_2 + \frac{3\lambda}{2} \|\theta^{\star}\|_1 \\ &\leq \frac{3\lambda\sqrt{s}}{2\kappa_{\min}^{\star}} \|\theta^{\star} - \hat{\theta}_S^{\tau}\|_{\mathcal{F}^{\star}} + \frac{3\lambda}{2} \|\theta^{\star}\|_1 \\ &\leq \frac{18\sqrt{5}\lambda^2 s \kappa_{\max}^{\star}}{\kappa_{\min}^{\star}} + \frac{3\lambda}{2} \|\theta^{\star}\|_1 \end{split}$$

where the last step is using (13). Hence we see that

$$\mathcal{L}(\tilde{\theta}) - \mathcal{L}(\theta^{\star}) \leq \mathcal{L}(\tilde{\theta}) - \mathcal{L}(\hat{\theta}_{S}^{\tau}) + \mathcal{L}(\hat{\theta}_{S}^{\tau}) - \mathcal{L}(\theta^{\star}) \leq \frac{581\kappa_{\max}^{\star}{}^{2}s\lambda^{2}}{\kappa_{\min}^{\star}} + \frac{3\lambda}{2}\|\theta^{\star}\|_{1}$$

Hence by condition (5) on λ we see that the pre-condition of the Theorem 3.4 is satisfied and hence we see that

$$\frac{1}{4} \|\tilde{\theta} - \theta^{\star}\|_{\mathcal{F}^{\star}}^{2} \leq \mathcal{L}(\tilde{\theta}) - \mathcal{L}(\theta^{\star}) \leq \mathcal{L}(\tilde{\theta}) - \mathcal{L}(\hat{\theta}_{S}^{\tau}) + \mathcal{L}(\hat{\theta}_{S}^{\tau}) - \mathcal{L}(\theta^{\star})$$

$$\leq \mathcal{L}(\tilde{\theta}) - \mathcal{L}(\hat{\theta}_{S}^{\tau}) + \frac{540\kappa_{\max}^{\star}^{2}s\lambda^{2}}{\kappa_{\min}^{\star}^{4}}$$

$$\leq \frac{3\lambda}{2} \|\tilde{\theta} - \hat{\theta}_{S}^{\tau}\|_{1} + \frac{540\kappa_{\max}^{\star}^{2}s\lambda^{2}}{\kappa_{\min}^{\star}^{4}}$$

$$\leq 6\lambda \|\tilde{\theta}_{S} - \hat{\theta}_{S}^{\tau}\|_{1} + \frac{540\kappa_{\max}^{\star}^{2}s\lambda^{2}}{\kappa_{\min}^{\star}^{4}}$$

$$\leq 6\lambda \sqrt{s} \|\tilde{\theta}_{S} - \hat{\theta}_{S}^{\tau}\|_{1} + \frac{540\kappa_{\max}^{\star}^{2}s\lambda^{2}}{\kappa_{\min}^{\star}^{\star}}$$

$$\leq 6\lambda \sqrt{s} \|\tilde{\theta}_{S} - \hat{\theta}_{S}^{\tau}\|_{1} + \frac{540\kappa_{\max}^{\star}^{2}s\lambda^{2}}{\kappa_{\min}^{\star}^{\star}}$$

$$\leq \frac{6\lambda \sqrt{s}}{\kappa_{\min}^{\star}} \|\tilde{\theta} - \hat{\theta}_{S}^{\tau}\|_{\mathcal{F}^{\star}} + \frac{540\kappa_{\max}^{\star}^{2}s\lambda^{2}}{\kappa_{\min}^{\star}^{\star}}$$

$$\leq \frac{6\lambda \sqrt{s}}{\kappa_{\min}^{\star}} \|\tilde{\theta} - \hat{\theta}_{S}^{\star}\|_{\mathcal{F}^{\star}} + \frac{6\lambda \sqrt{s}}{\kappa_{\min}^{\star}} \|\theta^{\star} - \hat{\theta}_{S}^{\tau}\|_{\mathcal{F}^{\star}} + \frac{540\kappa_{\max}^{\star}^{2}s\lambda^{2}}{\kappa_{\min}^{\star}}$$

$$\leq \frac{6\lambda \sqrt{s}}{\kappa_{\min}^{\star}} \|\tilde{\theta} - \theta^{\star}\|_{\mathcal{F}^{\star}} + \frac{161\kappa_{\max}^{\star}s\lambda^{2}}{\kappa_{\min}^{\star}} + \frac{540\kappa_{\max}^{\star}^{2}s\lambda^{2}}{\kappa_{\min}^{\star}}$$

$$\leq \frac{6\lambda \sqrt{s}}{\kappa_{\min}^{\star}} \|\tilde{\theta} - \theta^{\star}\|_{\mathcal{F}^{\star}} + \frac{161\kappa_{\max}^{\star}s\lambda^{2}}{\kappa_{\min}^{\star}} + \frac{540\kappa_{\max}^{\star}^{2}s\lambda^{2}}{\kappa_{\min}^{\star}}$$

$$(17)$$

Where (14) is obtained by applying Lemma 5.3 on $\Theta = \Theta_{\tau}$ and (15) is by Lemma 5.4 with $\Theta = \Theta_{\tau}$. (16) is by Assumption 4.1 and (17) is due to (13). Simplifying we conclude that

$$\frac{1}{4} \|\tilde{\theta} - \theta^{\star}\|_{\mathcal{F}^{\star}}^{2} \le \mathcal{L}(\tilde{\theta}) - \mathcal{L}(\theta^{\star}) \le \frac{6\lambda\sqrt{s}}{\kappa_{\min}^{\star}} \|\tilde{\theta} - \theta^{\star}\|_{\mathcal{F}^{\star}} + \frac{701\kappa_{\max}^{\star}^{2}s\lambda^{2}}{\kappa_{\min}^{\star}}$$
(18)

By the inequality that for any $a, b \in \mathbb{B}$, $ab \leq \frac{a^2}{2} + \frac{b^2}{2}$ we have

$$\frac{1}{2} \|\tilde{\theta} - \theta^{\star}\|_{\mathcal{F}^{\star}}^{2} \leq \frac{288\lambda^{2}s}{\kappa_{\min}^{\star}^{2}} + \frac{2804\kappa_{\max}^{\star}^{2}s\lambda^{2}}{\kappa_{\min}^{\star}^{4}}$$

Thus

$$\|\tilde{\theta} - \theta^*\|_{\mathcal{F}^*} \le \frac{24\lambda\sqrt{s}}{\kappa_{\min}^*} + \frac{75\kappa_{\max}^*\lambda\sqrt{s}}{\kappa_{\min}^*}^2$$

Using this in (18)

$$\mathcal{L}(\tilde{\theta}) - \mathcal{L}(\theta^*) \le \frac{144\lambda^2 s}{\kappa_{\min}^*} + \frac{450\kappa_{\max}^* \lambda^2 s}{\kappa_{\min}^*} + \frac{701\kappa_{\max}^* ^2 s \lambda^2}{\kappa_{\min}^*}$$

Simplifying we get the second claim of the theorem for the second case.

6 Errata

6.1 Cumulants of the Bernoulli Distribution

In Section 3.2.1, we claimed that the cumulants $c_k(\theta^*)$ of the Bernoulli distribution satisfy $c_k(\theta^*) \le c_2(\theta^*) = m_2(\theta^*)$ for $k \ge 3$. This claim is incorrect. Thanks to Francis Bach for pointing this out to us (personal email communication, 2015). However, all we needed was the existence of what we call an *analytic standardized cumulant*. The following lemma suffices to prove that one exists for the Bernoulli distribution. The result below is very likely to be classical. In any case, it follows easily from classical results on cumulants. We provide a proof below for completeness.

Lemma 6.1. The cumulants of the Bernoulli distribution satisfy, for $k \geq 3$:

$$|c_k(\theta^*)| \le (k-1)! \cdot c_2(\theta^*).$$

Proof. Let us work with the mean parameter $p = m_1(\theta^*)$. It is well known³ that

$$c_{k+1}(p) = p \cdot (1-p) \cdot c'_k(p)$$

where $c'_k(p)$ is the derivative of $c_k(p)$ w.r.t. p.

We first prove, by induction on $k \ge 2$, that $c_k(p)$ is a polynomial of degree k in p with k real roots in the interval [0,1], two of which are 0 and 1. Claim is true for k=2 since $c_2(p)=p(1-p)$. If $c_k(p)$ has k reals roots in [0,1] then $c_k'(p)$ has k-1 reals roots in [0,1]. This is because, by the Gauss-Lucas Theorem, roots of the derivative of a polynomial are in the convex hull of the roots of the polynomial itself. This immediately implies that $c_{k+1}(p)=p(1-p)c_k'(p)$ has k+1 real roots in [0,1] given that the extra factor p(1-p) has roots 0 and 1.

Given the claim above, we can express $c_k(p)$ as

$$c_k(p) = a_k \cdot p \cdot (1-p) \cdot \prod_{i=1}^{k-2} (p-r_i)$$

for some $a_k \in \mathbb{R}$ and $r_i \in [0,1]$. Note that $a_2 = 1$ and because $c_{k+1}(p) = p(1-p)c_k'(p)$, we also have $|a_{k+1}| = k|a_k|$. Therefore, $|a_k| = (k-1)!$ for all $k \ge 2$. The lemma now follows because

$$|c_k(p)| = |a_k| \cdot p \cdot (1-p) \cdot \left| \prod_{i=1}^{k-2} (p-r_i) \right|$$

$$\leq (k-1)! \cdot c_2(p) \cdot \prod_{i=1}^{k-2} |p-r_i|$$

$$\leq (k-1)! \cdot c_2(p).$$

Note that since $p, r_i \in [0, 1]$, we have $|p - r_i| \le 1$.

³See, for example, Eq. (3.3.12) on p. 56 of the book *Introduction to Statistical Inference* (Dover Publications, 1995) by E. S. Keeping.