# Robust Matrix Decomposition with Outliers 

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#### Abstract

Suppose a given observation matrix can be decomposed as the sum of a low-rank matrix and a sparse matrix (outliers), and the goal is to recover these individual components from the observed sum. Such additive decompositions have applications in a variety of numerical problems including system identification, latent variable graphical modeling, and principal components analysis. We study conditions under which recovering such a decomposition is possible via a combination of $\ell_{1}$ norm and trace norm minimization. We are specifically interested in the question of how many outliers are allowed so that convex programming can still achieve accurate recovery, and we obtain stronger recovery guarantees than previous studies. Moreover, we do not assume that the spatial pattern of outliers is random, which stands in contrast to related analyses under such assumptions via matrix completion.


## 1 Introduction

This work studies additive decompositions of matrices into sparse (outliers) and low-rank components. Such decompositions have found applications in a variety of numerical problems, including system identification (Chandrasekaran et al., 2009), latent variable graphical modeling (Chandrasekaran et al., 2010), and principal component analysis (Candès et al., 2009). In these settings, the user has an input matrix $Y \in \mathbb{R}^{m \times n}$ which is believed to be the sum of a sparse matrix $X_{S}$ and a low-rank matrix $X_{L}$. For instance, in the application to principal component analysis, $X_{L}$ represents a matrix of $m$ data points from a low-dimensional subspace of $\mathbb{R}^{n}$, and is corrupted by a sparse matrix $X_{S}$ of errors before being observed as

$$
Y=\underset{\substack{\text { (sparse) }}}{X_{S}}+\underset{\substack{\text { (low-rank) }}}{X_{L}}
$$

The goal is to recover the original data matrix $X_{L}$ (and the error components $X_{S}$ ) from the corrupted observations $Y$. In the latent variable model application of Chandrasekaran et al. (2010), $Y$ represents the precision matrix over visible nodes of a Gaussian graphical model, and $X_{S}$ represents the precision matrix over the visible nodes when conditioned on the hidden nodes. In general, $Y$ may be dense as a result of dependencies between visible nodes through the hidden nodes. However, $X_{S}$ will be sparse when the visible nodes are mostly independent after conditioning on the hidden nodes, and the difference $X_{L}=Y-X_{S}$ will be low-rank when the number of hidden nodes is small. The goal is then to infer the relevant dependency structure from just the visible nodes and measurements of their correlations.

Even if the matrix $Y$ is exactly the sum of a sparse matrix $X_{S}$ and a low-rank matrix $X_{L}$, it may be impossible to identify these components from the sum. For instance, the sparse matrix $X_{S}$ may be low-rank, or the low-rank matrix $X_{L}$ may be sparse. In such cases, these components may be confused for each other, and thus the desired decomposition of $Y$ may not be identifiable. Therefore, one must impose conditions on the sparse and low-rank components in order to guarantee their identifiability from $Y$.

We present sufficient conditions under which $X_{S}$ and $X_{L}$ are identifiable from the sum $Y$. Essentially, we require that $X_{S}$ not be too dense in any single row or column, and that the singular vectors of $X_{L}$ not be too sparse. The level of denseness and sparseness are considered jointly in the conditions in order to obtain the weakest possible conditions. Under a mild strengthening of the condition, we also show that $X_{S}$ and $X_{L}$ can be recovered by solving certain convex programs, and that the solution is robust under small perturbations of $Y$. The first program we consider is

$$
\min \lambda\left\|X_{S}\right\|_{\operatorname{vec}(1)}+\left\|X_{L}\right\|_{*}
$$

(subject to certain feasibility constraints such as $\left\|X_{S}+X_{L}-Y\right\| \leq \epsilon$ ) where $\|\cdot\|_{\mathrm{vec}(1)}$ is the entry-wise 1-norm and $\|\cdot\|_{*}$ is the trace norm. These norms are natural convex surrogates for the sparsity of $X_{S}$ and the rank of $X_{L}$ (Tibshirani, 1996; Fazel, 2002), which are generally intractable to optimize. We also considered a regularized formulation

$$
\min \frac{1}{2 \mu}\left\|X_{S}+X_{L}-Y\right\|_{\mathrm{vec}(2)}^{2}+\lambda\left\|X_{S}\right\|_{\mathrm{vec}(1)}+\left\|X_{L}\right\|_{*}
$$

where $\|\cdot\|_{\mathrm{vec}(2)}$ is the Frobenius norm; such a formulation may be more suitable in certain applications and enjoys different recovery guarantees.

### 1.1 Related work

Our work closely follows that of Chandrasekaran et al. (2009), who initiated the study of rank-sparsity incoherence and its application to matrix decompositions. There, the authors identify parameters that characterize the incoherence of $X_{S}$ and $X_{L}$ sufficient to guarantee identifiability and recovery using convex programs. However, their analysis of this characterization yields conditions that are significantly stronger than those given in the our present work. For instance, the allowed fraction of non-zero entries in $X_{S}$ is quickly vanishing as a function of the matrix size, even under the most favorable conditions on $X_{L}$; our analysis does not have this restriction and allows $X_{S}$ to have up to $\Omega(m n)$ non-zero entries when $X_{L}$ is lowrank and has non-sparse singular vectors. Therefore, for instance, in the application to principal component analysis, our analysis allows for up to a constant fraction of the data matrix entries to be corrupted by noise of arbitrary magnitude, whereas the analysis of Chandrasekaran et al. requires that it decrease as a function of the matrix dimensions. Moreover, Chandrasekaran et al. only consider exact decompositions, which may be unrealistic in certain applications; we allow for approximate decompositions, and study the effect of perturbations on the accuracy of the recovered components.

The application to principal component analysis with gross sparse errors was studied by Candès et al. (2009), building on previous results and analysis techniques for the related matrix completion problem (e.g., Candès and Recht 2009; Gross 2009). The sparse errors model studied by Candès et al. requires that the support of the sparse matrix $X_{S}$ be random, which can be unrealistic in some settings. However, the conditions are significantly weaker than those of Chandrasekaran et al. (2009): for instance, they allow for $\Omega(m n)$ non-zero entries in $X_{S}$. Our work makes no probabilistic assumption on the sparsity pattern of $X_{S}$ and instead studies purely deterministic structural conditions. The price we pay, however, is roughly a factor of $\operatorname{rank}\left(X_{L}\right)$ in what is allowed for the support size of $X_{S}$ (relative to the probabilistic analysis of Candès et al.). Narrowing this gap with alternative deterministic conditions is an interesting open problem. Follow-up work to (Candès et al., 2009) studies the robustness of the recovery procedure (Zhou et al., 2010), as well as quantitatively weaker conditions on $X_{S}$ (Ganesh et al., 2010), but these works are only considered under the random support model. Our work is therefore largely complementary to these probabilistic analyses.

### 1.2 Outline

We describe our main results in Section 2, In Section 3, we review a number of technical tools such as matrix and operator norms that are used to characterize the rank-sparsity incoherence properties of the desired decomposition. Section 4 analyses these incoherence properties in detail, giving sufficient conditions
for identifiability as well as for certifying the (approximate) optimality of a target decomposition for our optimization formulations. The main recovery guarantees are proved in Sections 5 and 6 ,

## 2 Main results

Fix an observation matrix $Y \in \mathbb{R}^{m \times n}$. Our goal is to (approximately) decompose the matrix $Y$ into the sum of a sparse matrix $X_{S}$ and a low-rank matrix $X_{L}$.

### 2.1 Optimization formulations

We consider two convex optimization problems over $\left(X_{S}, X_{L}\right) \in \mathbb{R}^{m \times n} \times \mathbb{R}^{m \times n}$. The first is the constrained formulation (parametrized by $\lambda>0, \epsilon_{\operatorname{vec}(1)} \geq 0$, and $\epsilon_{*} \geq 0$ )

$$
\begin{array}{cl}
\text { min } & \lambda\left\|X_{S}\right\|_{\operatorname{vec}(1)}+\left\|X_{L}\right\|_{*} \\
\text { s.t. } & \left\|X_{S}+X_{L}-Y\right\|_{\operatorname{vec}(1)} \leq \epsilon_{\operatorname{vec}(1)}  \tag{1}\\
& \left\|X_{S}+X_{L}-Y\right\|_{*} \leq \epsilon_{*}
\end{array}
$$

where $\|\cdot\|_{\text {vec (1) }}$ is the entry-wise 1 -norm, and $\|\cdot\|_{*}$ is the trace norm (i.e., sum of singular values). The second is the regularized formulation (with regularization parameter $\mu>0$ )

$$
\begin{equation*}
\min \quad \frac{1}{2 \mu}\left\|X_{S}+X_{L}-Y\right\|_{\mathrm{vec}(2)}^{2}+\lambda\left\|X_{S}\right\|_{\mathrm{vec}(1)}+\left\|X_{L}\right\|_{*} \tag{2}
\end{equation*}
$$

where $\|\cdot\|_{\operatorname{vec}(2)}$ is the Frobenius norm (entry-wise 2-norm).
We also consider adding a constraint to control $\left\|X_{L}\right\|_{\mathrm{vec}(\infty)}$, the entry-wise $\infty$-norm of $X_{L}$. To (11), we add the constraint

$$
\left\|X_{L}\right\|_{\mathrm{vec}(\infty)} \leq b
$$

and to (2), we add

$$
\left\|X_{S}-Y\right\|_{\operatorname{vec}(\infty)} \leq b
$$

The parameter $b$ is intended as a natural bound for $X_{L}$ and is typically known in applications. For example, in image processing, the values of interest may lie in the interval [ 0,255 ], and hence, we may take $b=500$ as a relaxation of the box constraint [0,255]. The core of our analyses do not rely on these additional constraints; we only consider them to obtain improved robustness guarantees for recovering $X_{L}$, which may be important in some applications.

### 2.2 Identifiability conditions

Our first result is a refinement of the rank-sparsity incoherence notion developed by Chandrasekaran et al. (2009). We characterize a target decomposition of $Y$ into $Y=\bar{X}_{S}+\bar{X}_{L}$ by the projection operators to subspaces associated with $\bar{X}_{S}$ and $\bar{X}_{L}$. Let

$$
\bar{\Omega}=\Omega\left(\bar{X}_{S}\right):=\left\{X \in \mathbb{R}^{m \times n}: \operatorname{supp}(X) \subseteq \operatorname{supp}\left(\bar{X}_{S}\right)\right\}
$$

be the space of matrices whose supports are subsets of the support of $\bar{X}_{S}$, and let $\mathcal{P}_{\bar{\Omega}}$ be the orthogonal projector to $\bar{\Omega}$ under the inner product $\langle A, B\rangle=\operatorname{tr}\left(A^{\top} B\right)$, where $\mathcal{P}_{\bar{\Omega}}(M)$ is given by

$$
\left[\mathcal{P}_{\bar{\Omega}}(M)\right]_{i, j}=\left\{\begin{array}{cl}
M_{i, j} & \text { if }(i, j) \in \operatorname{supp}\left(\bar{X}_{S}\right) \\
0 & \text { otherwise }
\end{array} \quad \forall i \in[m], j \in[n]\right.
$$

Furthermore, let

$$
\bar{T}=T\left(\bar{X}_{L}\right):=\left\{X_{1}+X_{2} \in \mathbb{R}^{m \times n}: \operatorname{range}\left(X_{1}\right) \subseteq \operatorname{range}\left(\bar{X}_{L}\right), \operatorname{range}\left(X_{2}^{\top}\right) \subseteq \operatorname{range}\left(\bar{X}_{L}^{\top}\right)\right\}
$$

be the span of matrices either in the row-space or column-space of $\bar{X}_{L}$, and let $\mathcal{P}_{\bar{T}}$ be the orthogonal projector to $\bar{T}$, again, under the inner product $\langle A, B\rangle=\operatorname{tr}\left(A^{\top} B\right)$; this is given by

$$
\mathcal{P}_{\bar{T}}(M)=\bar{U} \bar{U}^{\top} M+M \bar{V} \bar{V}^{\top}-\bar{U} \bar{U}^{\top} M \bar{V} \bar{V}^{\top}
$$

where $\bar{U} \in \mathbb{R}^{m \times \bar{r}}$ and $\bar{V} \in \mathbb{R}^{n \times \bar{r}}$ are, respectively, matrices of left and right orthonormal singular vectors corresponding to the non-zero singular values of $\bar{X}_{L}$, and $\bar{r}$ is the rank of $\bar{X}_{L}$. We will see that certain operator norms of $\mathcal{P}_{\bar{\Omega}}$ and $\mathcal{P}_{\bar{T}}$ can be bounded in terms of structural properties of $\bar{X}_{S}$ and $\bar{X}_{L}$. The first property measures the maximum number of non-zero entries in any row or column of $\bar{X}_{S}$ :

$$
\alpha(\rho):=\max \left\{\rho\left\|\operatorname{sign}\left(\bar{X}_{S}\right)\right\|_{1 \rightarrow 1}, \rho^{-1}\left\|\operatorname{sign}\left(\bar{X}_{S}\right)\right\|_{\infty \rightarrow \infty}\right\}
$$

where $\|M\|_{p \rightarrow q}:=\max \left\{\|M v\|_{q}: v \in \mathbb{R}^{n},\|v\|_{p} \leq 1\right\}$,

$$
\operatorname{sign}(M)_{i, j}=\left\{\begin{array}{rl}
-1 & \text { if } M_{i, j}<0 \\
0 & \text { if } M_{i, j}=0 \\
+1 & \text { if } M_{i, j}>0
\end{array} \quad \forall i \in[m], j \in[n]\right.
$$

and $\rho>0$ is a balancing parameter to accommodate disparity between the number of rows and columns; a natural choice for the balancing parameter is $\rho:=\sqrt{n / m}$. We remark that $\rho$ is only a parameter for the analysis; the optimization formulations do not directly involve $\rho$. Note that $\bar{X}_{S}$ may have $\Omega(m n)$ non-zero entries and $\alpha(\sqrt{n / m})=O(\sqrt{m n})$ as long as the non-zero entries of $\bar{X}_{S}$ are spread out over the entire matrix. Conversely, a sparse matrix with just $O(m+n)$ could have $\alpha(\sqrt{n / m})=\sqrt{m n}$ by having all of its non-zero entries in just a few rows and columns.

The second property measures the sparseness of the singular vectors of $\bar{X}_{L}$ :

$$
\beta(\rho):=\rho^{-1}\left\|\bar{U} \bar{U}^{\top}\right\|_{\operatorname{vec}(\infty)}+\rho\left\|\bar{V} \bar{V}^{\top}\right\|_{\operatorname{vec}(\infty)}+\|\bar{U}\|_{2 \rightarrow \infty}\|\bar{V}\|_{2 \rightarrow \infty}
$$

For instance, if the singular vectors of $\bar{X}_{L}$ are perfectly aligned with the coordinate axes, then $\beta(\rho)=$ $\Omega(1)$. On the other hand, if the left and right singular vectors have entries bounded by $\sqrt{c / m}$ and $\sqrt{c / n}$, respectively, for some $c \geq 1$, then $\beta(\sqrt{n / m}) \leq 3 c \bar{r} / \sqrt{m n}$.

Our main identifiability result is the following.
Theorem 1. If $\inf _{\rho>0} \alpha(\rho) \beta(\rho)<1$, then $\bar{\Omega} \cap \bar{T}=\{0\}$.
Theorem 1 is an immediate consequence of the following lemma (also given as Lemma 10).
Lemma 1. For all $M \in \mathbb{R}^{m \times n},\left\|\mathcal{P}_{\bar{\Omega}}\left(\mathcal{P}_{\bar{T}}(M)\right)\right\|_{\text {vec }(1)} \leq \inf _{\rho>0} \alpha(\rho) \beta(\rho)\|M\|_{\text {vec }(1)}$.
Proof of Theorem 11. Take any $M \in \bar{\Omega} \cap \bar{T}$. By Lemma 1 , $\left\|\mathcal{P}_{\bar{\Omega}}\left(\mathcal{P}_{\bar{T}}(M)\right)\right\|_{\operatorname{vec}(1)} \leq \alpha(\rho) \beta(\rho)\|M\|_{\text {vec }(1)}$. On the other hand, $\mathcal{P}_{\bar{\Omega}}\left(\mathcal{P}_{\bar{T}}(M)\right)=M$, so $\alpha(\rho) \beta(\rho)<1$ implies $\|M\|_{\text {vec }(1)}=0$, i.e., $M=0$.

Clearly, if $\bar{\Omega} \cap \bar{T}$ contains a matrix other than 0 , then $\left\{\left(\bar{X}_{S}+M, \bar{X}_{L}-M\right): M \in \bar{\Omega} \cap \bar{T}\right\}$ gives a family of sparse/low-rank decompositions of $Y=\bar{X}_{S}+\bar{X}_{L}$ with at least the same sparsity and rank as $\left(\bar{X}_{S}, \bar{X}_{L}\right)$. Conversely, if $\bar{\Omega} \cap \bar{T}=\{0\}$, then any matrix in the direct sum $\bar{\Omega} \oplus \bar{T}$ has exactly one decomposition into a matrix $A \in \bar{\Omega}$ plus a matrix $B \in \bar{T}$, and in this sense $\left(\bar{X}_{S}, \bar{X}_{L}\right)$ is identifiable.

Note that, as we have argued above, the condition $\inf _{\rho>0} \alpha(\rho) \beta(\rho)<1$ may be achieved even by matrices $\bar{X}_{S}$ with $\Omega(m n)$ non-zero entries, provided that the non-zero entries of $\bar{X}_{S}$ are sufficiently spread out, and that $\bar{X}_{L}$ is low-rank and has singular vectors far from the coordinate bases. This is in contrast with the conditions studied by Chandrasekaran et al. (2009). Their analysis uses a different characterization of $\bar{X}_{S}$ and $\bar{X}_{L}$, which leads to a stronger identifiability condition in certain cases. Roughly, if $\bar{X}_{S}$ has an approximately symmetric sparsity pattern (so $\left.\left\|\operatorname{sign}\left(\bar{X}_{S}\right)\right\|_{1 \rightarrow 1} \approx\left\|\operatorname{sign}\left(\bar{X}_{S}\right)\right\|_{\infty \rightarrow \infty}\right)$, then Chandrasekaran et al. require $\alpha(1) \sqrt{\beta(1)}<1$ for square $n \times n$ matrices ${ }^{1}$ Since $\beta(1)=\Omega(1 / n)$ for any $\bar{X}_{L} \in \mathbb{R}^{n \times n}$, the condition implies

[^0]$\alpha(1)^{2}=O(n)$. Therefore $\bar{X}_{S}$ must have at most $O(n)$ non-zero entries (or else $\alpha(1)^{2}$ becomes super-linear). In other words, the fraction of non-zero entries allowed in $\bar{X}_{S}$ by the condition $\alpha(1) \sqrt{\beta(1)}<1$ is quickly vanishing as a function of $n$.

### 2.3 Recovery guarantees

Our next results are guarantees on (approximately) recovering the sparse/low-rank decomposition ( $\bar{X}_{S}, \bar{X}_{L}$ ) from $Y=\bar{X}_{S}+\bar{X}_{L}$ via solving either convex optimization problems (11) or (21). We require a mild strengthening of the condition $\inf _{\rho>0} \alpha(\rho) \beta(\rho)<1$, as well as appropriate settings of $\lambda>0$ and $\mu>0$ for our recovery guarantees. Before continuing, we first define another property of $\bar{X}_{L}$ :

$$
\gamma:=\left\|\bar{U} \bar{V}^{\top}\right\|_{\operatorname{vec}(\infty)}
$$

which is approximately the same as (in fact, bounded above by) the third term in the definition of $\beta(\rho)$. The quantities $\alpha(\rho), \beta(\rho)$, and $\gamma$ are central to our analysis. Therefore we state the following proposition for reference, which provides a more intuitive understanding of their behavior. This is the only part in which explicit dimensional dependencies comes into our analysis.
Proposition 1. Let $m_{0}$ the maximum number of non-zero entries of $\bar{X}_{S}$ per column and $n_{0}$ be the maximum number of non-zero entries of $\bar{X}_{S}$ per row. Let $\bar{r}$ be the rank of $\bar{U}$ and $\bar{V}$. Assume further that $m_{0} \leq c_{1} m / \bar{r}$ and $n_{0} \leq c_{1} n / \bar{r}$ for some $c_{1} \in(0,1)$, and $\|\bar{U}\|_{\operatorname{vec}(\infty)} \leq \sqrt{c_{2} / m}$ and $\|\bar{V}\|_{\operatorname{vec}(\infty)} \leq \sqrt{c_{2} / n}$ for some $c_{2}>0$. Then with $\rho=\sqrt{n / m}$, we have

$$
\alpha(\rho) \leq \frac{c_{1}}{\bar{r}} \sqrt{m n}, \quad \beta(\rho) \leq \frac{3 c_{2} \bar{r}}{\sqrt{m n}}, \quad \gamma \leq \frac{c_{2} \bar{r}}{\sqrt{m n}}
$$

We now proceed with conditions for the regularized formulation (2). Let $E:=Y-\left(\bar{X}_{S}+\bar{X}_{L}\right)$ and

$$
\begin{aligned}
\epsilon_{2 \rightarrow 2} & :=\|E\|_{2 \rightarrow 2} \\
\epsilon_{\mathrm{vec}(\infty)} & :=\|E\|_{\operatorname{vec}(\infty)}+\left\|\mathcal{P}_{\bar{T}}(E)\right\|_{\operatorname{vec}(\infty)} .
\end{aligned}
$$

We require the following, for some $\rho>0$ and $c>1$ :

$$
\begin{align*}
& \alpha(\rho) \beta(\rho)<1  \tag{3}\\
& \lambda \leq \frac{(1-\alpha(\rho) \beta(\rho))\left(1-c \cdot \mu^{-1} \epsilon_{2 \rightarrow 2}\right)-c \cdot \alpha(\rho) \mu^{-1} \epsilon_{\mathrm{vec}(\infty)}-c \cdot \alpha(\rho) \gamma}{c \cdot \alpha(\rho)}  \tag{4}\\
& \lambda \geq c \cdot \frac{\gamma+\mu^{-1}(2-\alpha(\rho) \beta(\rho)) \epsilon_{\mathrm{vec}(\infty)}}{1-\alpha(\rho) \beta(\rho)-c \cdot \alpha(\rho) \beta(\rho)}>0 . \tag{5}
\end{align*}
$$

For instance, if for some $\rho>0$,

$$
\begin{equation*}
\alpha(\rho) \gamma \leq \frac{1}{41} \quad \text { and } \quad \alpha(\rho) \beta(\rho) \leq \frac{3}{41} \tag{6}
\end{equation*}
$$

then the conditions are satisfied for $c=2$ provided that $\mu$ and $\lambda$ are chosen to satisfy

$$
\begin{equation*}
\mu \geq \max \left\{4 \cdot \epsilon_{2 \rightarrow 2}, \frac{2}{15} \cdot \frac{\epsilon_{\mathrm{vec}(\infty)}}{\lambda}\right\} \quad \text { and } \quad \frac{15}{2} \cdot \gamma \leq \lambda \leq \frac{15}{82} \cdot \frac{1}{\alpha(\rho)} \tag{7}
\end{equation*}
$$

Note that (6) can be satisfied when $c_{1} \leq c_{2}^{-1} / 41$ in Proposition 1 .
For the constrained formulation (11), our analysis requires the same conditions as above, except with $E$ set to 0 . Note that our analysis still allows for approximate decompositions; it is only the conditions that are formulated with $E=0$. Specifically, we require for some $\rho>0$ and $c>1$ :

$$
\begin{align*}
& \alpha(\rho) \beta(\rho)<1  \tag{8}\\
& \lambda \leq \frac{1-\alpha(\rho) \beta(\rho)-c \cdot \alpha(\rho) \gamma}{c \cdot \alpha(\rho)}  \tag{9}\\
& \lambda \geq c \cdot \frac{\gamma}{1-\alpha(\rho) \beta(\rho)-c \cdot \alpha(\rho) \beta(\rho)}>0 \tag{10}
\end{align*}
$$

For instance, if for some $\rho>0$,

$$
\begin{equation*}
\alpha(\rho) \gamma \leq \frac{1}{15} \quad \text { and } \quad \alpha(\rho) \beta(\rho) \leq \frac{1}{5} \tag{11}
\end{equation*}
$$

then the conditions are satisfied for $c=2$ provided that $\lambda$ is chosen to satisfy

$$
\begin{equation*}
5 \gamma \leq \lambda \leq \frac{1}{3 \alpha(\rho)} \tag{12}
\end{equation*}
$$

Note that (11) can be satisfied when $c_{1} \leq c_{2}^{-1} / 15$ in Proposition 1.
In summary, Proposition 1 shows that our results can be applied even with $m_{0}=\Omega(m / \bar{r})$ and $n_{0}=$ $\Omega(n / \bar{r})$ outliers. In contrast, the results of Chandrasekaran et al. (2009) only apply under the condition $\max \left(m_{0}, n_{0}\right)=O(\sqrt{\min (m, n) / \bar{r}})$, which is significantly stronger. Moreover, unlike the analysis of Candès et al. (2009), we do not have to assume that $\operatorname{supp}\left(\bar{X}_{S}\right)$ is random.

The following theorem gives our recovery guarantee for the constrained formulation (11).
Theorem 2. Fix a target pair $\left(\bar{X}_{S}, \bar{X}_{L}\right) \in \mathbb{R}^{m \times n} \times \mathbb{R}^{m \times n}$ satisfying $\left\|Y-\left(\bar{X}_{S}+\bar{X}_{L}\right)\right\|_{\operatorname{vec}(1)} \leq \epsilon_{\mathrm{vec}(1)}$ and $\left\|Y-\left(\bar{X}_{S}+\bar{X}_{L}\right)\right\|_{*} \leq \epsilon_{*}$. Assume the conditions (8), (9), and (10) hold for some $\rho>0$ and $c>1$. Let $\left(\hat{X}_{S}, \hat{X}_{L}\right) \in \mathbb{R}^{m \times n}$ be the solution to the convex optimization problem (11). We have

$$
\begin{aligned}
\max \left\{\left\|\hat{X}_{S}-\bar{X}_{S}\right\|_{\operatorname{vec}(1)}\right. & \left.,\left\|\hat{X}_{L}-\bar{X}_{L}\right\|_{\operatorname{vec}(1)}\right\} \\
\leq & \left(1+(1-1 / c)^{-1} \cdot \frac{2-\alpha(\rho) \beta(\rho)}{1-\alpha(\rho) \beta(\rho)}\right) \cdot \epsilon_{\mathrm{vec}(1)}+(1-1 / c)^{-1} \cdot \frac{2-\alpha(\rho) \beta(\rho)}{1-\alpha(\rho) \beta(\rho)} \cdot \epsilon_{*} / \lambda
\end{aligned}
$$

If, in addition for some $b \geq\left\|\bar{X}_{L}\right\|_{\mathrm{vec}(\infty)}$, either:

- the optimization problem (11) is augmented with the constraint $\left\|X_{L}\right\|_{\operatorname{vec}(\infty)} \leq b$, or
- $\hat{X}_{L}$ is post-processed by replacing $\left[\hat{X}_{L}\right]_{i, j}$ with $\min \left\{\max \left\{\left[\hat{X}_{L}\right]_{i, j},-b\right\}, b\right\}$ for all $i, j$,
then we also have

$$
\left\|\hat{X}_{L}-\bar{X}_{L}\right\|_{\mathrm{vec}(2)} \leq \min \left\{\left\|\hat{X}_{L}-\bar{X}_{L}\right\|_{\mathrm{vec}(1)}, \sqrt{2 b \cdot\left\|\hat{X}_{L}-\bar{X}_{L}\right\|_{\mathrm{vec}(1)}}\right\}
$$

The proof of Theorem 2 is in Section 5. It is clear that if $Y=\bar{X}_{S}+\bar{X}_{L}$, then we can set $\epsilon_{\operatorname{vec}(1)}=\epsilon_{*}=0$ and we obtain exact recovery: $\hat{X}_{S}=\bar{X}_{S}$ and $\hat{X}_{L}=\bar{X}_{L}$. Moreover, any perturbation $Y-\left(\bar{X}_{S}+\bar{X}_{L}\right)$ affects the accuracy of $\left(\hat{X}_{S}, \hat{X}_{L}\right)$ in entry-wise 1-norm by an amount $O\left(\epsilon_{\text {vec }(1)}+\epsilon_{*} / \lambda\right)$. Note that here, the parameter $\lambda$ serves to balance the entry-wise 1-norm and trace norm of the perturbation in the same way it is used in the objective function of (11). So, for instance, if we have the simplified conditions (11), then we may choose $\lambda=\sqrt{(5 / 3) \gamma / \alpha(\rho)}$ to satisfy (12), upon which the error bound becomes

$$
\max \left\{\left\|\hat{X}_{S}-\bar{X}_{S}\right\|_{\mathrm{vec}(1)},\left\|\hat{X}_{L}-\bar{X}_{L}\right\|_{\mathrm{vec}(1)}\right\}=O\left(\epsilon_{\mathrm{vec}(1)}+\sqrt{\frac{\alpha(\rho)}{\gamma}} \cdot \epsilon_{*}\right)
$$

It is possible to modify the constraints in (11) to use norms other than $\|\cdot\|_{\text {vec }(1)}$ and $\|\cdot\|_{*}$; the analysis could at the very least be modified by simply using standard relationships to change between norms, although this may introduce new slack in the bounds. Finally, the second part of the theorem shows how the accuracy of $\hat{X}_{L}$ in Frobenius norm can be improved by adding an additional constraint or by post-processing the solution.

Now we state our recovery guarantees for the regularized formulation (21).

Theorem 3. Fix a target pair $\left(\bar{X}_{S}, \bar{X}_{L}\right) \in \mathbb{R}^{m \times n} \times \mathbb{R}^{m \times n}$. Let $E:=Y-\left(\bar{X}_{S}+\bar{X}_{L}\right)$ and

$$
\begin{aligned}
\epsilon_{2 \rightarrow 2} & :=\|E\|_{2 \rightarrow 2} \\
\epsilon_{\mathrm{vec}(\infty)} & :=\|E\|_{\operatorname{vec}(\infty)}+\left\|\mathcal{P}_{\bar{T}}(E)\right\|_{\operatorname{vec}(\infty)} \\
\epsilon_{*}^{\prime} & :=\left\|\mathcal{P}_{\bar{T}}(E)\right\|_{*} .
\end{aligned}
$$

Let $\bar{k}:=\left|\operatorname{supp}\left(\bar{X}_{S}\right)\right|$ and $\bar{r}:=\operatorname{rank}\left(\bar{X}_{L}\right)$. Assume the conditions (3), (4), and (5) hold for some $\rho>0$ and $c>1$. Let $\left(\hat{X}_{S}, \hat{X}_{L}\right) \in \mathbb{R}^{m \times n}$ be the solution to the convex optimization problem (2) augmented with the constraint $\left\|X_{S}-Y\right\|_{\operatorname{vec}(\infty)} \leq b$ for some $b \geq\left\|\bar{X}_{S}-Y\right\|_{\operatorname{vec}(\infty)} \quad(b=\infty$ is allowed). Let

$$
\begin{aligned}
\bar{r}^{\prime}:= & \left(\lambda+\mu^{-1} \epsilon_{\mathrm{vec}(\infty)}\right) \cdot \frac{2 \bar{k}}{1-\alpha(\rho) \beta(\rho)} \cdot\left(\lambda+\gamma+\mu^{-1} \epsilon_{\mathrm{vec}(\infty)}\right) \\
& +\left(1+2 \mu^{-1} \epsilon_{2 \rightarrow 2}\right) \cdot 2 \bar{r} \cdot\left(\frac{2 \alpha(\rho)}{1-\alpha(\rho) \beta(\rho)} \cdot\left(\lambda+\gamma+\mu^{-1} \epsilon_{\mathrm{vec}(\infty)}\right)+1+2 \mu^{-1} \epsilon_{2 \rightarrow 2}\right)
\end{aligned}
$$

We have

$$
\begin{aligned}
\left\|\hat{X}_{S}-\bar{X}_{S}\right\|_{\operatorname{vec}(1)} & \leq \frac{\bar{r}^{\prime} \cdot(1-1 / c)^{-1} \lambda^{-1} \cdot \mu+\lambda \bar{k} \cdot \mu+2 \sqrt{k \bar{r}} \cdot \mu+\bar{k} \cdot \epsilon_{\mathrm{vec}(\infty)}}{1-\alpha(\rho) \beta(\rho)} \\
\left\|\hat{X}_{S}-\bar{X}_{S}\right\|_{\operatorname{vec}(2)} & \leq \min \left\{\left\|\hat{X}_{S}-\bar{X}_{S}\right\|_{\operatorname{vec}(1)}, \sqrt{2 b \cdot\left\|\hat{X}_{S}-\bar{X}_{S}\right\|_{\operatorname{vec}(1)}}\right\} \\
\left\|\hat{X}_{L}-\bar{X}_{L}\right\|_{*} & \leq \sqrt{2 \bar{r}} \cdot\left\|\hat{X}_{S}-\bar{X}_{S}\right\|_{\operatorname{vec}(2)}+\epsilon_{*}^{\prime}+\left(\frac{\bar{r}^{\prime} \cdot(1-1 / c)^{-1}}{2}+2 \bar{r}\right) \cdot \mu
\end{aligned}
$$

The proof of Theorem 3 is in Section 6. As before, if $Y=\bar{X}_{S}+\bar{X}_{L}$ so $E=0$, then we can set $\mu \rightarrow 0$ and obtain exact recovery with $\hat{X}_{S}=\bar{X}_{S}$ and $\hat{X}_{L}=\bar{X}_{L}$. When the perturbation $E$ is non-zero, we control the accuracy of $\bar{X}_{S}$ in entry-wise 1-norm and 2-norm, and the accuracy of $\bar{X}_{L}$ in trace norm. Under the simplified conditions (6), we can choose $\lambda=(15 / 82) / \alpha(\rho)$ and $\mu=\max \left\{4 \epsilon_{2 \rightarrow 2}, 2 \epsilon_{\mathrm{vec}(\infty)} /(15 \lambda)\right\}$ to satisfy (77); this leads to the error bounds

$$
\begin{aligned}
& \left\|\hat{X}_{S}-\bar{X}_{S}\right\|_{\mathrm{vec}(1)}=O\left(\bar{r} \alpha(\rho) \cdot \max \left\{\epsilon_{2 \rightarrow 2}, \alpha(\rho) \epsilon_{\mathrm{vec}(\infty)}\right\}\right) \\
& \left\|\hat{X}_{L}-\bar{X}_{L}\right\|_{*}=O\left(\sqrt{\bar{r}} \cdot \min \left\{\sqrt{b \cdot\left\|\hat{X}_{S}-\bar{X}_{S}\right\|_{\operatorname{vec}(1)}},\left\|\hat{X}_{S}-\bar{X}_{S}\right\|_{\operatorname{vec}(1)}\right\}+\epsilon_{*}^{\prime}+\bar{r} \cdot \max \left\{\epsilon_{2 \rightarrow 2}, \alpha(\rho) \epsilon_{\mathrm{vec}(\infty)}\right\}\right)
\end{aligned}
$$

(here, we have used the facts $\bar{k} \leq \alpha(\rho)^{2}, \alpha(\rho) \lambda=\Theta(1)$, and $\bar{r}^{\prime}=O(\bar{r})$, which also implies that $\bar{k} \cdot \epsilon_{\mathrm{vec}(\infty)}=$ $\left.O\left(\alpha(\rho) \cdot \alpha(\rho) \epsilon_{\mathrm{vec}(\infty)}\right)\right)$. Finally, note that if the constraint $\left\|X_{S}-Y\right\|_{\mathrm{vec}(\infty)} \leq b$ is added (i.e., $b<\infty$ ), then the requirement $b \geq\left\|\bar{X}_{S}-Y\right\|_{\operatorname{vec}(\infty)}$ can be satisfied with $b:=\left\|\bar{X}_{S}\right\|_{\operatorname{vec}(\infty)}+\epsilon_{\operatorname{vec}(\infty)}$. This allows for a possibly improved bound on $\left\|\hat{X}_{L}-\bar{X}_{L}\right\|_{*}$.

### 2.4 Examples

We illustrate our main results with some simple examples.

### 2.4.1 Random models

We first consider a random model for the matrices $\bar{X}_{S}$ and $\bar{X}_{L}$ (Chandrasekaran et al., 2009). Let the support of $\bar{X}_{S}$ be chosen uniformly at random $\tilde{k}$ times over the $[m] \times[n]$ matrix entries (so that one entry can be selected multiple times). The value of the entries in the chosen support can be arbitrary. With high probability, we have

$$
\left\|\operatorname{sign}\left(\bar{X}_{S}\right)\right\|_{1 \rightarrow 1}=O\left(\frac{\tilde{k} \log n}{n}\right) \quad \text { and } \quad\left\|\operatorname{sign}\left(\bar{X}_{S}\right)\right\|_{\infty \rightarrow \infty}=O\left(\frac{\tilde{k} \log m}{m}\right)
$$

so for $\rho:=\sqrt{(n \log m) /(m \log n)}$, we have

$$
\alpha(\rho)=O\left(\tilde{k} \sqrt{\frac{(\log m)(\log n)}{m n}}\right)
$$

The logarithmic factors are due to collisions in the random process. Now let $\bar{U}$ and $\bar{V}$ be chosen uniformly at random over all families of $\bar{r}$ orthonormal vectors in $\mathbb{R}^{m}$ and $\mathbb{R}^{n}$, respectively. Using arguments similar to those in (Candès and Recht, 2009), one can show that with high probability,

$$
\begin{array}{ll}
\left\|\bar{U} \bar{U}^{\top}\right\|_{\operatorname{vec}(\infty)}=O\left(\frac{\bar{r} \log m}{m}\right) & \left\|\bar{V} \bar{V}^{\top}\right\|_{\operatorname{vec}(\infty)}=O\left(\frac{\bar{r} \log n}{n}\right) \\
\|\bar{U}\|_{2 \rightarrow \infty}=O\left(\sqrt{\frac{\bar{r} \log m}{m}}\right) & \|\bar{V}\|_{2 \rightarrow \infty}=O\left(\sqrt{\frac{\bar{r} \log n}{n}}\right)
\end{array}
$$

so for the previously chosen $\rho$, we have

$$
\beta(\rho)=O\left(\bar{r} \sqrt{\frac{(\log m)(\log n)}{m n}}\right) \quad \text { and } \quad \gamma=O\left(\bar{r} \sqrt{\frac{(\log m)(\log n)}{m n}}\right) .
$$

Therefore

$$
\alpha(\rho) \beta(\rho)=O\left(\frac{\tilde{k} \bar{r}(\log m)(\log n)}{m n}\right) \quad \text { and } \quad \alpha(\rho) \gamma=O\left(\frac{\tilde{k} \bar{r}(\log m)(\log n)}{m n}\right)
$$

both of which are $\ll 1$ provided that

$$
\tilde{k} \leq \delta \cdot \frac{m n}{\bar{r}(\log m)(\log n)}
$$

for a small enough constant $\delta \in(0,1)$. In other words, when $\bar{X}_{L}$ is low-rank, the matrix $\bar{X}_{S}$ can have nearly a constant fraction of its entries be non-zero while still allowing for exact decomposition of $Y=\bar{X}_{S}+\bar{X}_{L}$. Our guarantee improves over that of Chandrasekaran et al. (2009) by roughly a factor of $\Omega\left((m n)^{1 / 4}\right)$, but is worse by a factor of $\bar{r}(\log m)(\log n)$ relative to the guarantees of Candès et al. (2009) for the random model. Therefore there is a gap between our generic deterministic analysis and a direct probabilistic analysis of this random model, and this gap seems unavoidable with sparsity conditions based on $\alpha(\rho)$. It is an interesting open problem to find alternative characterizations of $\operatorname{supp}\left(\bar{X}_{S}\right)$ that can narrow or close this gap.

### 2.4.2 Principal component analysis with outliers

Suppose $\bar{X}_{L}$ is matrix of $m$ data points lying in a low-dimensional subspace of $\mathbb{R}^{n}$, and $Z$ is a random matrix with independent Gaussian noise entries with variance $\sigma^{2}$. Then $Y^{\prime}=\bar{X}_{L}+Z$ is the standard model for principal component analysis. We augment the model with a sparse noise component $\bar{X}_{S}$ to obtain $Y=\bar{X}_{S}+\bar{X}_{L}+Z$; here, we allow the non-zero entries of $\bar{X}_{S}$ to possibly approach infinity.

According to Theorem [3, we need to estimate $\|Z\|_{2 \rightarrow 2},\|Z\|_{\text {vec }(\infty)},\left\|\mathcal{P}_{\bar{T}}(Z)\right\|_{\text {vec }(\infty)}$, and $\left\|\mathcal{P}_{\bar{T}}(Z)\right\|_{*}$. We have the following with high probability (Davidson and Szarek, 2001),

$$
\|Z\|_{2 \rightarrow 2} \leq \sigma \sqrt{m}+\sigma \sqrt{n}+O(\sigma)
$$

Using standard arguments with the rotational invariance of the Gaussian distribution, we also have

$$
\|Z\|_{\operatorname{vec}(\infty)} \leq O(\sigma \log (m n)) \quad \text { and } \quad\left\|\mathcal{P}_{\bar{T}}(Z)\right\|_{\operatorname{vec}(\infty)} \leq O(\sigma \log (m n))
$$

with high probability. Finally, by Lemma 5, we have

$$
\left\|\mathcal{P}_{\bar{T}}(Z)\right\|_{*} \leq 2 \bar{r}\|Z\|_{2 \rightarrow 2} \leq 2 \bar{r} \sigma \sqrt{m}+2 \bar{r} \sigma \sqrt{n}+O(\bar{r} \sigma)
$$

Suppose $\left(\bar{X}_{S}, \bar{X}_{L}\right)$ has $\alpha(\rho) \leq c_{1}(\sqrt{m n} / \bar{r}), \beta(\rho)=\Theta(\bar{r} / \sqrt{m n})$, and $\gamma=\Theta(\bar{r} / \sqrt{m n})$ and satisfies the simplified condition (6). This can be achieved with $c_{1} c_{2} \leq 1 / 41$ in Proposition (1). Also assume $\lambda$ and $\mu$ are chosen to satisfy (77), and that $b \geq\left\|\bar{X}_{L}\right\|_{\mathrm{vec}(\infty)}+\epsilon_{\mathrm{vec}(\infty)}$. Then we note that $\bar{k}=O\left(c_{1}^{2} m n / \bar{r}^{2}\right)$, and thus have from Theorem 3 (see the discussion thereafter):

$$
\begin{aligned}
\left\|\hat{X}_{S}-\bar{X}_{S}\right\|_{\mathrm{vec}(1)} & =O\left(c_{1} \sqrt{m n} \max \{\sigma \sqrt{m}+\sigma \sqrt{n}, \sigma \sqrt{m n} \log (m n) / \bar{r}\}\right) \\
& =O\left(\sigma c_{1} m n \log (m n) / \bar{r}\right) \\
\left\|\hat{X}_{L}-\bar{X}_{L}\right\|_{*} & \left.=O\left(\sqrt{b \sigma c_{1} m n \log (m n) / \bar{r}}+\bar{r} \sigma(\sqrt{m}+\sqrt{n})\right)+c_{1} \sqrt{m n}\right)
\end{aligned}
$$

where we may take $\left.b=O\left(\sigma \log (m n)+\left\|X_{L}\right\|_{\mathrm{vec}(\infty)}\right)\right)$.
Now consider the situation where both $m, n \rightarrow \infty$, and assume that $\left\|\bar{X}_{L}\right\|_{\mathrm{vec}(\infty)}$ remains bounded. If $c_{1}(\log (m n))^{2}=o(1)$, which implies that the we have at most $o\left(m /(\log (m n))^{2}\right)$ outliers per column and $o\left(n /(\log (m n))^{2}\right)$ outliers per row, then

$$
\left\|\hat{X}_{L}-\bar{X}_{L}\right\|_{*}=O(\bar{r} \sigma(\sqrt{m}+\sqrt{n}))
$$

That is, the normalized trace norm $\left\|\hat{X}_{L}-\bar{X}_{L}\right\|_{*} / \sqrt{n m} \rightarrow 0$. This means that we can correctly recover the principal components of $\bar{X}_{L}$ with both outliers and random noise, when both $m$ and $n$ are large and $c_{1}(\log (m n))^{2}=o(1)$ in Proposition 1

## 3 Technical preliminaries

### 3.1 Norms, inner products, and projections

Our analysis involves a variety of norms of vectors, matrices (viewed as elements of a vector space as well as linear operators of vectors), and linear operators of matrices; we define these and related notions in this section.

### 3.1.1 Entry-wise norms

For any $p \in[1, \infty]$, define $\|v\|_{p}:=\left(\sum_{i}\left|v_{i}\right|^{p}\right)^{1 / p}$ be the $p$-norm of a vector $v$ (with $\left.\|v\|_{\infty}:=\max _{i}\left|v_{i}\right|\right)$. Also, define $\|M\|_{\text {vec }(p)}:=\left(\sum_{i, j}\left|M_{i, j}\right|^{p}\right)^{1 / p}$ to be the entry-wise $p$-norm of a matrix $M$ (again, with $\|M\|_{\text {vec }(\infty)}:=$ $\left.\max _{i, j}\left|M_{i, j}\right|\right)$. Note that $\|\cdot\|_{\mathrm{vec}(2)}$ corresponds to the Frobenius norm.

### 3.1.2 Inner products, linear operators, and orthogonal projections

We endow $\mathbb{R}^{m \times n}$ with the inner product $\langle\cdot, \cdot\rangle$ between matrices that induces the Frobenius norm $\|\cdot\|_{\operatorname{vec}(2)}$; this is given by $\langle M, N\rangle=\operatorname{tr}\left(M^{\top} N\right)$.

For a linear operator $\mathcal{T}: \mathbb{R}^{m \times n} \rightarrow \mathbb{R}^{m \times n}$, we denote its adjoint by $\mathcal{T}^{*}$; this is the unique linear operator that satisfies $\left\langle\mathcal{T}^{*}(M), N\right\rangle=\langle M, \mathcal{T}(N)\rangle$ for all $M \in \mathbb{R}^{m \times n}$ and $N \in \mathbb{R}^{m \times n}$ (in this work, we only consider bounded linear operators). For any two linear operators $\mathcal{T}_{1}$ and $\mathcal{T}_{2}$, we let $\mathcal{T}_{1} \circ \mathcal{T}_{2}$ denote their composition as defined by $\left(\mathcal{T}_{1} \circ \mathcal{T}_{2}\right)(M):=\mathcal{T}_{1}\left(\mathcal{T}_{2}(M)\right)$.

Given a subspace $W \subseteq \mathbb{R}^{m \times n}$, we let $W^{\perp}$ denote its orthogonal complement, and let $\mathcal{P}_{W}: \mathbb{R}^{m \times n} \rightarrow \mathbb{R}^{m \times n}$ denote the orthogonal projector to $W$ with respect to $\langle\cdot, \cdot\rangle$, i.e., the unique linear operator with range $W$ and satisfying $\mathcal{P}_{W}{ }^{*}=\mathcal{P}_{W}$ and $\mathcal{P}_{W} \circ \mathcal{P}_{W}=\mathcal{P}_{W}$.

### 3.1.3 Induced norms

For any two vector norms $\|\cdot\|_{p}$ and $\|\cdot\|_{q}$, define $\|M\|_{p \rightarrow q}:=\max _{x \neq 0}\|M x\|_{q} /\|x\|_{p}$ to be the corresponding induced operator norm of a matrix $M$. Our analysis uses the following special cases which have alternative definitions:

- $\|M\|_{1 \rightarrow 1}=\max _{j}\left\|M e_{j}\right\|_{1}$,
- $\|M\|_{1 \rightarrow 2}=\max _{j}\left\|M e_{j}\right\|_{2}$,
- $\|M\|_{2 \rightarrow 2}=$ spectral norm of $M$ (i.e., largest singular value of $M$ ),
- $\|M\|_{2 \rightarrow \infty}=\max _{i}\left\|M^{\top} e_{i}\right\|_{2}$, and
- $\|M\|_{\infty \rightarrow \infty}=\max _{i}\left\|M^{\top} e_{i}\right\|_{1}$.

Here, $e_{i}$ is the $i$ th coordinate vector which has a 1 in the $i$ th position and 0 elsewhere.
Finally, we also consider induced operator norms of linear matrix operators $\mathcal{T}: \mathbb{R}^{m \times n} \rightarrow \mathbb{R}^{m \times n}$ (in particular, projection operators with respect to $\langle\cdot, \cdot\rangle$ ). For any two matrix norms $\|\cdot\|_{\diamond}$ and $\|\cdot\|_{\odot}$, define $\|\mathcal{T}\|_{\diamond \rightarrow \odot}:=\max _{M \neq 0}\|\mathcal{T}(M)\|_{\odot} /\|M\|_{\diamond}$.

### 3.1.4 Other norms

The trace norm (or nuclear norm) $\|M\|_{*}$ of a matrix $M$ is the sum of the singular values of $M$. We will also make use of a hybrid matrix norm $\|\cdot\|_{\sharp(\rho)}$, parametrized by $\rho>0$, which we define by

$$
\|M\|_{\sharp(\rho)}:=\max \left\{\rho\|M\|_{1 \rightarrow 1}, \rho^{-1}\|M\|_{\infty \rightarrow \infty}\right\} .
$$

Also define $\|M\|_{b(\rho)}:=\sup _{\|N\|_{\sharp(\rho)} \leq 1}\langle M, N\rangle$, i.e., the dual of $\|\cdot\|_{\sharp(\rho)}$ (see below).

### 3.1.5 Dual pairs

The matrix norm $\|\cdot\|_{\odot}$ is said to be dual to $\|\cdot\|_{\bullet}$ if, for all $M \in \mathbb{R}^{m \times n},\|M\|_{\odot}=\sup _{\|N\|_{\bullet} \leq 1}\langle M, N\rangle$.
Proposition 2. Fix any matrix norm $\|\cdot\|_{\oplus}$, and let $\|\cdot\|_{\odot}$ be its dual. For all $M \in \mathbb{R}^{m \times n}$ and $N \in \mathbb{R}^{m \times n}$, we have

$$
\langle M, N\rangle \leq\|M\|_{\bullet}\|N\|_{\odot} .
$$

Proposition 3. Fix any any linear matrix operator $\mathcal{T}: \mathbb{R}^{m \times n} \rightarrow \mathbb{R}^{m \times n}$ and any pair of matrix norms $\| \cdot$ and $\|\cdot\|_{\boldsymbol{\&}}$. We have

$$
\|\mathcal{T}\|_{\boldsymbol{\omega} \rightarrow \boldsymbol{\omega}}=\left\|\mathcal{T}^{*}\right\|_{\diamond \rightarrow \infty}
$$

where $\|\cdot\|_{\odot}$ is dual to $\|\cdot\|_{\uparrow}$, and $\|\cdot\|_{\diamond}$ is dual to $\|\cdot\|$
The following pairs of matrix norms are dual to each other:

1. $\|\cdot\|_{\operatorname{vec}(p)}$ and $\|\cdot\|_{\operatorname{vec}(q)}$ where $1 / p+1 / q=1$;
2. $\|\cdot\|_{*}$ and $\|\cdot\|_{2 \rightarrow 2}$;
3. $\|\cdot\|_{\sharp(\rho)}$ and $\|\cdot\|_{b(\rho)}$ (by definition).

### 3.1.6 Some lemmas

First we show that the $\|\cdot\|_{\sharp(\rho)}$ norm (for any $\rho>0$ ) bounds the spectral norm $\|\cdot\|_{2 \rightarrow 2}$.
Lemma 2. For any $M \in \mathbb{R}^{m \times n}$, we have for all $\rho>0$,

$$
\|M\|_{2 \rightarrow 2} \leq\|M\|_{\sharp(\rho)}
$$

Proof. Let $\sigma$ be the largest singular value of $M$, and let $u \in \mathbb{R}^{m}$ and $v \in \mathbb{R}^{n}$ be, respectively, associated left and right singular vectors. Then

$$
\left\|\left[\begin{array}{cc}
0 & \rho M \\
\rho^{-1} M^{\top} & 0
\end{array}\right]\left[\begin{array}{c}
\rho^{1 / 2} u \\
\rho^{-1 / 2} v
\end{array}\right]\right\|_{1}=\left\|\left[\begin{array}{c}
\rho^{1 / 2} M v \\
\rho^{-1 / 2} M^{\top} u
\end{array}\right]\right\|_{1}=\sigma\left\|\left[\begin{array}{c}
\rho^{1 / 2} u \\
\rho^{-1 / 2} v
\end{array}\right]\right\|_{1} .
$$

Moreover, by definition of $\|\cdot\|_{1 \rightarrow 1}$,

$$
\left\|\left[\begin{array}{cc}
0 & \rho M \\
\rho^{-1 / 2} M^{\top} & 0
\end{array}\right]\left[\begin{array}{c}
\rho^{1 / 2} u \\
\rho^{-1 / 2} v
\end{array}\right]\right\|_{1} \leq\left\|\left[\begin{array}{cc}
0 & \rho M \\
\rho^{-1} M^{\top} & 0
\end{array}\right]\right\|_{1 \rightarrow 1}\left\|\left[\begin{array}{c}
\rho^{1 / 2} u \\
\rho^{-1 / 2} v
\end{array}\right]\right\|_{1}
$$

Therefore

$$
\begin{aligned}
\|M\|_{2 \rightarrow 2}=\sigma & \leq\left\|\left[\begin{array}{cc}
0 & \rho M \\
\rho^{-1} M^{\top} & 0
\end{array}\right]\right\|_{1 \rightarrow 1} \\
& =\max \left\{\left\|\rho^{-1} M^{\top}\right\|_{1 \rightarrow 1},\|\rho M\|_{1 \rightarrow 1}\right\}=\max \left\{\rho^{-1}\|M\|_{\infty \rightarrow \infty}, \rho\|M\|_{1 \rightarrow 1}\right\}=\|M\|_{\sharp(\rho)}
\end{aligned}
$$

The following lemma is the dual of Lemma 2 .
Lemma 3. For any $M \in \mathbb{R}^{m \times n}$, we have for all $\rho>0$,

$$
\|M\|_{b(\rho)} \leq\|M\|_{*} .
$$

Proof. We know that $\|M\|_{b(\rho)}=\langle M, N\rangle$ for some matrix $N$ such that $\|N\|_{\sharp(\rho)}=1$. Therefore $\|N\|_{2 \rightarrow 2} \leq 1$ from Lemma 2, and thus using Proposition 2,

$$
\|M\|_{b(\rho)}=\langle M, N\rangle \leq\|M\|_{*}\|N\|_{2 \rightarrow 2} \leq\|M\|_{*}
$$

Finally we state a lemma concerning the invertibility of a certain block-form operator used in our analysis.
Lemma 4. Fix any matrix norm $\|\cdot\| \oplus$ on $\mathbb{R}^{m \times n}$ and linear operators $\mathcal{T}_{1}: \mathbb{R}^{m \times n} \rightarrow \mathbb{R}^{m \times n}$ and $\mathcal{T}_{2}: \mathbb{R}^{m \times n} \rightarrow$ $\mathbb{R}^{m \times n}$. Let $\mathcal{I}: \mathbb{R}^{m \times n} \rightarrow \mathbb{R}^{m \times n}$ be the identity operator, and suppose $\left\|\mathcal{T}_{1} \circ \mathcal{T}_{2}\right\|_{\boldsymbol{\bullet} \rightarrow \boldsymbol{\oplus}}<1$.

1. $\mathcal{I}-\mathcal{T}_{1} \circ \mathcal{T}_{2}$ is invertible and satisfies

$$
\left\|\left(\mathcal{I}-\mathcal{T}_{1} \circ \mathcal{T}_{2}\right)^{-1}\right\|_{\boldsymbol{\omega} \rightarrow \boldsymbol{\omega}} \leq \frac{1}{1-\left\|\mathcal{T}_{1} \circ \mathcal{T}_{2}\right\|_{\boldsymbol{\omega} \rightarrow \boldsymbol{\omega}}}
$$

2. The linear operator on $\mathbb{R}^{m \times n} \times \mathbb{R}^{m \times n}$

$$
\left[\begin{array}{ll}
\mathcal{I} & \mathcal{T}_{1} \\
\mathcal{T}_{2} & \mathcal{I}
\end{array}\right]
$$

is invertible, and its inverse is given by

$$
\begin{aligned}
{\left[\begin{array}{cc}
\mathcal{I} & \mathcal{T}_{1} \\
\mathcal{T}_{2} & \mathcal{I}
\end{array}\right]^{-1} } & =\left[\begin{array}{cc}
\left(\mathcal{I}-\mathcal{T}_{1} \circ \mathcal{T}_{2}\right)^{-1} & -\left(\mathcal{I}-\mathcal{T}_{1} \circ \mathcal{T}_{2}\right)^{-1} \circ \mathcal{T}_{1} \\
-\mathcal{T}_{2} \circ\left(\mathcal{I}-\mathcal{T}_{1} \circ \mathcal{T}_{2}\right)^{-1} & \mathcal{I}+\mathcal{T}_{2} \circ\left(\mathcal{I}-\mathcal{T}_{1} \circ \mathcal{T}_{2}\right)^{-1} \circ \mathcal{T}_{1}
\end{array}\right] \\
& =\left[\begin{array}{cc}
\left(\mathcal{I}-\mathcal{T}_{1} \circ \mathcal{T}_{2}\right)^{-1} & -\left(\mathcal{I}-\mathcal{T}_{1} \circ \mathcal{T}_{2}\right)^{-1} \circ \mathcal{T}_{1} \\
-\left(\mathcal{I}-\mathcal{T}_{2} \circ \mathcal{T}_{1}\right)^{-1} \circ \mathcal{T}_{2} & \left(\mathcal{I}-\mathcal{T}_{2} \circ \mathcal{T}_{1}\right)^{-1}
\end{array}\right]
\end{aligned}
$$

Proof. The first claim is a standard application of Taylor expansions. The second claim then follows from formulae of block matrix inverses using Schur complements.

### 3.2 Projection operators and subdifferential sets

Recall the definitions of the following subspaces

$$
\Omega\left(X_{S}\right):=\left\{X \in \mathbb{R}^{m \times n}: \operatorname{supp}(X) \subseteq \operatorname{supp}\left(X_{S}\right)\right\}
$$

and

$$
T\left(X_{L}\right):=\left\{X_{1}+X_{2} \in \mathbb{R}^{m \times n}: \operatorname{range}\left(X_{1}\right) \subseteq \operatorname{range}\left(X_{L}\right), \operatorname{range}\left(X_{2}^{\top}\right) \subseteq \operatorname{range}\left(X_{L}^{\top}\right)\right\} .
$$

The orthogonal projectors to these spaces are given in the following proposition.
Proposition 4. Fix any $X_{S} \in \mathbb{R}^{m \times n}$ and $X_{L} \in \mathbb{R}^{m \times n}$. For any matrix $M \in \mathbb{R}^{m \times n}$,

$$
\left[\mathcal{P}_{\Omega\left(X_{S}\right)}(M)\right]_{i, j}=\left\{\begin{array}{cl}
M_{i, j} & \text { if }(i, j) \in \operatorname{supp}\left(X_{S}\right) \\
0 & \text { otherwise }
\end{array}\right.
$$

for all $1 \leq i \leq m$ and $1 \leq j \leq n$, and

$$
\mathcal{P}_{T\left(X_{L}\right)}(M)=U U^{\top} M+M V V^{\top}-U U^{\top} M V V^{\top}
$$

where $U$ and $V$ are the matrices of left and right singular vectors of $X_{L}$.
Lemma 5. Under the setting of Proposition 4.

$$
\begin{aligned}
\left\|\mathcal{P}_{\Omega\left(X_{S}\right)}(M)\right\|_{\operatorname{vec}(1)} & \leq \sqrt{\left|\operatorname{supp}\left(X_{S}\right)\right|}\left\|\mathcal{P}_{\Omega\left(X_{S}\right)}(M)\right\|_{\operatorname{vec}(2)} \leq \sqrt{\left|\operatorname{supp}\left(X_{S}\right)\right|}\|M\|_{\operatorname{vec}(2)} \\
\left\|\mathcal{P}_{\Omega\left(X_{S}\right)}(M)\right\|_{\operatorname{vec}(1)} & \leq\left|\operatorname{supp}\left(X_{S}\right)\left\|\mathcal{P}_{\Omega\left(X_{S}\right)}(M)\right\|_{\operatorname{vec}(\infty)} \leq\right| \operatorname{supp}\left(X_{S}\right)\|M\|_{\mathrm{vec}(\infty)} \\
\left\|\mathcal{P}_{T\left(X_{L}\right)}(M)\right\|_{2 \rightarrow 2} & \leq 2\|M\|_{2 \rightarrow 2} \\
\left\|\mathcal{P}_{T\left(X_{L}\right)}(M)\right\|_{*} & \leq 2 \operatorname{rank}\left(X_{L}\right)\|M\|_{2 \rightarrow 2} \\
\left\|\mathcal{P}_{T\left(X_{L}\right)}(M)\right\|_{\operatorname{vec}(2)} & \leq 2 \sqrt{\operatorname{rank}\left(X_{L}\right)\|M\|_{2 \rightarrow 2} .}
\end{aligned}
$$

Proof. The first and second claims rely on the fact that $\left|\operatorname{supp}\left(\mathcal{P}_{\Omega\left(X_{S}\right)}(M)\right)\right| \leq\left|\operatorname{supp}\left(X_{S}\right)\right|$, as well as the fact that $\mathcal{P}_{\Omega\left(X_{S}\right)}$ is an orthonormal projector with respect to the inner product that induces the $\|\cdot\|_{\mathrm{vec}(2)}$ norm. For the third claim, note that

$$
\left\|\mathcal{P}_{T\left(X_{L}\right)}(M)\right\|_{2 \rightarrow 2} \leq\left\|U U^{\top} M\right\|_{2 \rightarrow 2}+\left\|\left(I-U U^{\top}\right) M V V^{\top}\right\|_{2 \rightarrow 2} \leq 2\|M\|_{2 \rightarrow 2} .
$$

The remaining claims use a similar decomposition as the third claim as well as the fact that

$$
\max \left\{\operatorname{rank}\left(U U^{\top} M\right), \operatorname{rank}\left(\left(I-U U^{\top}\right) M V V^{\top}\right)\right\} \leq \operatorname{rank}\left(X_{L}\right) .
$$

Define

$$
\operatorname{sign}\left(X_{S}\right) \in\{-1,0,+1\}^{m \times n}
$$

to be the matrix whose $(i, j)$ th entry is $\operatorname{sign}\left(\left[X_{S}\right]_{i, j}\right)$, and define

$$
\operatorname{orth}\left(X_{L}\right):=U V^{\top},
$$

where $U$ and $V$, respectively, are matrices of the left and right orthonormal singular vectors of $X_{L}$ corresponding to non-zero singular values. The following proposition characterizes the subdifferential sets for the non-smooth norms $\|\cdot\|_{\text {vec(1) }}$ and $\|\cdot\|_{*}$ (Watson, 1992).
Proposition 5. The subdifferential set of $X_{S} \mapsto\left\|X_{S}\right\|_{\mathrm{vec}(1)}$ is

$$
\partial_{X_{S}}\left(\left\|X_{S}\right\|_{\operatorname{vec}(1)}\right)=\left\{G \in \mathbb{R}^{m \times n}:\|G\|_{\operatorname{vec}(\infty)} \leq 1, \mathcal{P}_{\Omega\left(X_{S}\right)}(G)=\operatorname{sign}\left(X_{S}\right)\right\} ;
$$

the subdifferential set of $X_{L} \mapsto\left\|X_{L}\right\|_{*}$ is

$$
\partial_{X_{L}}\left(\left\|X_{L}\right\|_{*}\right)=\left\{G \in \mathbb{R}^{m \times n}:\|G\|_{2 \rightarrow 2} \leq 1, \mathcal{P}_{T\left(X_{L}\right)}(G)=\operatorname{orth}\left(X_{L}\right)\right\} .
$$

The following lemma is a simple consequence of subgradient properties.
Lemma 6. Fix $\lambda>0$ and define the function $g\left(X_{S}, X_{L}\right):=\lambda\left\|X_{S}\right\|_{\operatorname{vec}(1)}+\left\|X_{L}\right\|_{*}$. Consider any $\left(\bar{X}_{S}, \bar{X}_{L}\right)$ in $\mathbb{R}^{m \times n} \times \mathbb{R}^{m \times n}$. If there exists $Q \in \mathbb{R}^{m \times n}$ such that: $Q$ is a subgradient of $\lambda\left\|X_{S}\right\|_{\text {vec }(1)}$ at $X_{S}=\bar{X}_{S}, Q$ is a subgradient of $\left\|X_{L}\right\|_{*}$ at $X_{L}=\bar{X}_{L}$, and $\left\|\mathcal{P}_{\Omega\left(\bar{X}_{S}\right)^{\perp}}(Q)\right\|_{\mathrm{vec}(\infty)} \leq \lambda / c$ and $\left\|\mathcal{P}_{T\left(\bar{X}_{L}\right)^{\perp}}(Q)\right\|_{2 \rightarrow 2} \leq 1 / c$ for some $c>1$, then
$g\left(X_{S}, X_{L}\right)-g\left(\bar{X}_{S}, \bar{X}_{L}\right) \geq\left\langle Q, X_{S}+X_{L}-\bar{X}_{S}-\bar{X}_{L}\right\rangle+(1-1 / c)\left(\lambda\left\|\mathcal{P}_{\bar{\Omega}^{\perp}}\left(X_{S}-\bar{X}_{S}\right)\right\|_{\operatorname{vec}(1)}+\left\|\mathcal{P}_{\bar{T}^{\perp}}\left(X_{L}-\bar{X}_{L}\right)\right\|_{*}\right)$
for all $\left(X_{S}, X_{L}\right) \in \mathbb{R}^{m \times n} \times \mathbb{R}^{m \times n}$.
Proof. Let $\bar{\Omega}:=\Omega\left(\bar{X}_{S}\right), \bar{T}:=T\left(\bar{X}_{L}\right), \Delta_{S}:=X_{S}-\bar{X}_{S}$, and $\Delta_{L}: X_{L}-\bar{X}_{L}$. For any subgradient $G \in$ $\partial_{X_{S}}\left(\lambda\left\|\bar{X}_{S}\right\|_{\text {vec }(1)}\right)$, we have $G-Q=\mathcal{P}_{\bar{\Omega}}(G)+\mathcal{P}_{\bar{\Omega}^{\perp}}(G)-\mathcal{P}_{\bar{\Omega}}(Q)-\mathcal{P}_{\bar{\Omega}^{\perp}}(Q)=\mathcal{P}_{\bar{\Omega}^{\top}}(G)-\mathcal{P}_{\bar{\Omega}^{\top}}(Q)$. Therefore

$$
\begin{aligned}
& \lambda\left\|\bar{X}_{S}+\Delta_{S}\right\|_{\operatorname{vec}(1)}-\lambda\left\|\bar{X}_{S}\right\|_{\operatorname{vec}(1)}-\left\langle Q, \Delta_{S}\right\rangle \\
& \geq \sup \left\{\left\langle G, \Delta_{S}\right\rangle-\left\langle Q, \Delta_{S}\right\rangle: G \in \partial_{X_{S}}\left(\lambda\left\|\bar{X}_{S}\right\|_{\operatorname{vec}(1)}\right)\right\} \\
& \geq \sup \left\{\left\langle G-Q, \Delta_{S}\right\rangle: G \in \partial_{X_{S}}\left(\lambda\left\|\bar{X}_{S}\right\|_{\operatorname{vec}(1)}\right)\right\} \\
& =\sup \left\{\left\langle\mathcal{P}_{\bar{\Omega}^{\perp}}(G)-\mathcal{P}_{\bar{\Omega}^{\perp}}(Q), \Delta_{S}\right\rangle: G \in \partial_{X_{S}}\left(\lambda\left\|\bar{X}_{S}\right\|_{\operatorname{vec}(1)}\right)\right\} \\
& =\sup \left\{\left\langle\mathcal{P}_{\bar{\Omega}^{\perp}}(G)-\mathcal{P}_{\bar{\Omega}^{\perp}}(Q), \mathcal{P}_{\bar{\Omega}^{\perp}}\left(\Delta_{S}\right)\right\rangle: G \in \partial_{X_{S}}\left(\lambda\left\|\bar{X}_{S}\right\|_{\operatorname{vec}(1)}\right)\right\} \\
& =\sup \left\{\left\langle\mathcal{P}_{\bar{\Omega}^{\perp}}(G), \mathcal{P}_{\bar{\Omega}^{\perp}}\left(\Delta_{S}\right)\right\rangle-\left\langle\mathcal{P}_{\bar{\Omega}^{\perp}}(Q), \mathcal{P}_{\bar{\Omega}^{\perp}}\left(\Delta_{S}\right)\right\rangle: G \in \partial_{X_{S}}\left(\lambda\left\|\bar{X}_{S}\right\|_{\operatorname{vec}(1)}\right)\right\} \\
& \left.=\lambda \| \mathcal{P}_{\bar{\Omega}^{\perp}} \Delta_{S}\right) \|_{\operatorname{vec}(1)}-\left\langle\mathcal{P}_{\bar{\Omega}^{\perp}}(Q), \mathcal{P}_{\bar{\Omega}^{\perp}}\left(\Delta_{S}\right)\right\rangle \\
& \left.\geq \lambda\left\|\mathcal{P}_{\bar{\Omega}^{\perp}}\left(\Delta_{S}\right)\right\|_{\operatorname{vec}(1)}-\left\|\mathcal{P}_{\bar{\Omega}^{\perp}}(Q)\right\|_{\operatorname{vec}(\infty)}\right)\left\|\mathcal{P}_{\bar{\Omega}^{\perp}}\left(\Delta_{S}\right)\right\|_{\operatorname{vec}(1)} \\
& \geq \lambda(1-1 / c)\left\|\mathcal{P}_{\bar{\Omega}^{\perp}}\left(\Delta_{S}\right)\right\|_{\operatorname{vec}(1)}
\end{aligned}
$$

where the second-to-last inequality uses the duality of $\|\cdot\|_{\operatorname{vec}(1)}$ and $\|\cdot\|_{\mathrm{vec}(\infty)}$ and Proposition 3. Similarly,

$$
\left\|\bar{X}_{L}-\Delta_{L}\right\|_{*}-\left\|\bar{X}_{L}\right\|_{*}-\left\langle Q, \Delta_{L}\right\rangle \geq(1-1 / c)\left\|\mathcal{P}_{\bar{T}^{\perp}}\left(\Delta_{L}\right)\right\|_{*}
$$

by noting the duality of $\|\cdot\|_{*}$ and $\|\cdot\|_{2 \rightarrow 2}$. Combining these gives the desired inequality.

## 4 Rank-sparsity incoherence

Throughout this section, we fix a target $\left(\bar{X}_{S}, \bar{X}_{L}\right) \in \mathbb{R}^{m \times n} \times \mathbb{R}^{m \times n}$, and let $\bar{\Omega}:=\Omega\left(\bar{X}_{S}\right)$ and $\bar{T}:=T\left(\bar{X}_{L}\right)$. Also let $\bar{U}$ and $\bar{V}$ be, respectively, matrices of the left and right singular vectors of $\bar{X}_{L}$ corresponding to non-zero singular values. Recall the following structural properties of $\bar{X}_{S}$ and $\bar{X}_{L}$ :

$$
\begin{aligned}
\alpha(\rho) & :=\left\|\operatorname{sign}\left(\bar{X}_{S}\right)\right\|_{\|(\rho)}=\max \left\{\rho\left\|\operatorname{sign}\left(\bar{X}_{S}\right)\right\|_{1 \rightarrow 1}, \rho^{-1}\left\|\operatorname{sign}\left(\bar{X}_{S}\right)\right\|_{\infty \rightarrow \infty}\right\} ; \\
\beta(\rho) & :=\rho^{-1}\left\|\bar{U} \bar{U}^{\top}\right\|_{\operatorname{vec}(\infty)}+\rho\left\|\bar{V} \bar{V}^{\top}\right\|_{\operatorname{vec}(\infty)}+\|\bar{U}\|_{2 \rightarrow \infty}\|\bar{V}\|_{2 \rightarrow \infty} ; \\
\gamma & :=\left\|\operatorname{orth}\left(\bar{X}_{L}\right)\right\|_{\operatorname{vec}(\infty)}=\left\|\bar{U} \bar{V}^{\top}\right\|_{\operatorname{vec}(\infty)} .
\end{aligned}
$$

The parameter $\rho$ is a balancing parameter to handle disparity between row and column dimensions. The quantity $\alpha(\rho)$ is the maximum number of non-zero entries in any single row or column. The quantities $\beta(\rho)$ and $\gamma$ measure the coherence of the singular vectors of $\bar{X}_{L}$, that is, the alignment of the singular vectors with the coordinate bases. For instance, under the conditions of Proposition we have (with $\rho=\sqrt{n / m}$ )

$$
\alpha(\rho) \leq c_{1} \sqrt{m n}, \quad \beta(\rho) \leq \frac{3 c_{2} \operatorname{rank}\left(\bar{X}_{L}\right)}{\sqrt{m n}} \quad \text { and } \quad \gamma \leq \frac{c_{2} \operatorname{rank}\left(\bar{X}_{L}\right)}{\sqrt{m n}}
$$

for some constants $c_{1}$ and $c_{2}$.

### 4.1 Operator norms of projection operators

We show that under the condition $\inf _{\rho>0} \alpha(\rho) \beta(\rho)<1$, the pair ( $\bar{X}_{S}, \bar{X}_{L}$ ) is identifiable from its sum $\bar{X}_{S}+\bar{X}_{L}$ (Theorem 1 . This is achieved by proving that the composition of projection operators $\mathcal{P}_{\bar{\Omega}}$ and $\mathcal{P}_{\bar{T}}$ is a contraction as per Lemma 1, which in turn implies that $\bar{\Omega} \cap \bar{T}=\{0\}$.

The following two lemmas bound the projection operators $\mathcal{P}_{\bar{\Omega}}$ and $\mathcal{P}_{\bar{T}}$ in complementary norms.
Lemma 7. For any $M \in \mathbb{R}^{m \times n}$ and $p \in\{1, \infty\}$, we have

$$
\left\|\mathcal{P}_{\bar{\Omega}}(M)\right\|_{p \rightarrow p} \leq\left\|\operatorname{sign}\left(\bar{X}_{S}\right)\right\|_{p \rightarrow p}\|M\|_{\operatorname{vec}(\infty)}
$$

This implies, for all $\rho>0$,

$$
\left\|\mathcal{P}_{\bar{\Omega}}\right\|_{\mathrm{vec}(\infty) \rightarrow \sharp(\rho)} \leq \alpha(\rho) .
$$

Proof. Define $s\left(X_{S}\right) \in\{0,1\}^{m \times n}$ to be the entry-wise absolute value of $\operatorname{sign}\left(X_{S}\right)$. We have

$$
\begin{aligned}
\left\|\mathcal{P}_{\bar{\Omega}}(M)\right\|_{p \rightarrow p} & =\max \left\{\left\|\mathcal{P}_{\bar{\Omega}}(M) v\right\|_{p}:\|v\|_{p} \leq 1\right\} \\
& \leq\left\|\mathcal{P}_{\bar{\Omega}}(M)\right\|_{\operatorname{vec}(\infty)} \max \left\{\left\|s\left(\mathcal{P}_{\bar{\Omega}}(M)\right) v\right\|_{p}:\|v\|_{p} \leq 1\right\} \\
& \leq\|M\|_{\operatorname{vec}(\infty)} \max \left\{\left\|s\left(\bar{X}_{S}\right) v\right\|_{p}:\|v\|_{p} \leq 1\right\} \\
& =\|M\|_{\operatorname{vec}(\infty)}\left\|\operatorname{sign}\left(\bar{X}_{S}\right)\right\|_{p \rightarrow p} .
\end{aligned}
$$

The second part follows from the definitions of $\|\cdot\|_{\sharp(\rho)}$ and $\alpha(\rho)$.
Lemma 8. For any $M \in \mathbb{R}^{m \times n}$, we have

$$
\left\|\mathcal{P}_{\bar{T}}(M)\right\|_{\operatorname{vec}(\infty)} \leq\left\|\bar{U} \bar{U}^{\top}\right\|_{\operatorname{vec}(\infty)}\|M\|_{1 \rightarrow 1}+\left\|\bar{V} \bar{V}^{\top}\right\|_{\operatorname{vec}(\infty)}\|M\|_{\infty \rightarrow \infty}+\|\bar{U}\|_{2 \rightarrow \infty}\|\bar{V}\|_{2 \rightarrow \infty}\|M\|_{2 \rightarrow 2}
$$

This implies, for all $\rho>0$,

$$
\left\|\mathcal{P}_{\bar{T}}\right\|_{\sharp(\rho) \rightarrow \operatorname{vec}(\infty)} \leq \beta(\rho) .
$$

Proof. We have $\left\|\mathcal{P}_{\bar{T}}(M)\right\|_{\operatorname{vec}(\infty)}=\left\|\bar{U} \bar{U}^{\top} M+M \bar{V} \bar{V}^{\top}-\bar{U} \bar{U}^{\top} M \bar{V} \bar{V}^{\top}\right\|_{\operatorname{vec}(\infty)} \leq\left\|\bar{U} \bar{U}^{\top} M\right\|_{\operatorname{vec}(\infty)}+\left\|M \bar{V} \bar{V}^{\top}\right\|_{\operatorname{vec}(\infty)}+$ $\left\|\bar{U} \bar{U}^{\top} M \bar{V} \bar{V}^{\top}\right\|_{\mathrm{vec}(\infty)}$ by the triangle inequality. The bounds for each term now follow from the definitions:

$$
\begin{aligned}
\left\|\bar{U} \bar{U}^{\top} M\right\|_{\operatorname{vec}(\infty)} & =\max _{i}\left\|M^{\top} \bar{U} \bar{U}^{\top} e_{i}\right\|_{\infty} \\
& \leq\left\|M^{\top}\right\|_{\infty \rightarrow \infty} \max _{i}\left\|\bar{U} \bar{U}^{\top} e_{i}\right\|_{\infty} \\
& =\|M\|_{1 \rightarrow 1}\left\|\bar{U} \bar{U}^{\top}\right\|_{\operatorname{vec}(\infty)} \\
\left\|M \bar{V} \bar{V}^{\top}\right\|_{\operatorname{vec}(\infty)} & =\max _{j}\left\|M \bar{V} \bar{V}^{\top} e_{j}\right\|_{\infty} \\
& \leq\|M\|_{\infty \rightarrow \infty} \max _{j}\left\|\bar{V} \bar{V}^{\top} e_{j}\right\|_{\infty} \\
& =\|M\|_{\infty \rightarrow \infty}\|\bar{V} \bar{V}\|_{\operatorname{vec}(\infty)}
\end{aligned}
$$

and

$$
\begin{aligned}
\left\|\bar{U} \bar{U}^{\top} M \bar{V} \bar{V}^{\top}\right\|_{\operatorname{vec}(\infty)} & =\max _{i, j}\left|e_{i}^{\top} \bar{U}\left(\bar{U}^{\top} M \bar{V}\right) \bar{V}^{\top} e_{j}\right| \\
& \leq \max _{i, j}\left\|\bar{U}^{\top} e_{i}\right\|_{2}\left\|\bar{U}^{\top} M \bar{V}\right\|_{2 \rightarrow 2}\left\|\bar{V}^{\top} e_{j}\right\|_{2} \quad \text { (Cauchy-Schwarz) } \\
& \leq\|M\|_{2 \rightarrow 2}\|\bar{U}\|_{2 \rightarrow \infty}\|\bar{V}\|_{2 \rightarrow \infty} \\
& \leq\|M\|_{\sharp(\rho)}\|\bar{U}\|_{2 \rightarrow \infty}\|\bar{V}\|_{2 \rightarrow \infty} \quad \text { (Lemma (2). }
\end{aligned}
$$

The second part now follows the definition of $\beta(\rho)$.

Now we show that the composition of $\mathcal{P}_{\bar{\Omega}}$ and $\mathcal{P}_{\bar{T}}$ gives a contraction under the certain norms and their duals.
Lemma 9. For all $\rho>0$,

1. $\left\|\mathcal{P}_{\bar{\Omega}} \circ \mathcal{P}_{\bar{T}}\right\|_{\sharp(\rho) \rightarrow \sharp(\rho)} \leq \alpha(\rho) \beta(\rho)$;
2. $\left\|\mathcal{P}_{\bar{T}} \circ \mathcal{P}_{\bar{\Omega}}\right\|_{\operatorname{vec}(\infty) \rightarrow \operatorname{vec}(\infty)} \leq \alpha(\rho) \beta(\rho)$;

Proof. Immediate from Lemma 7 and Lemma 8 .
Lemma 10. For all $\rho>0$,

1. $\left\|\mathcal{P}_{\bar{T}} \circ \mathcal{P}_{\bar{\Omega}}\right\|_{b(\rho) \rightarrow b(\rho)} \leq \alpha(\rho) \beta(\rho) ;$
2. $\left\|\mathcal{P}_{\bar{\Omega}} \circ \mathcal{P}_{\bar{T}}\right\|_{\operatorname{vec}(1) \rightarrow \operatorname{vec}(1)} \leq \alpha(\rho) \beta(\rho)$.

Proof. First note that $\left(\mathcal{P}_{\bar{T}} \circ \mathcal{P}_{\bar{\Omega}}\right)^{*}=\mathcal{P}_{\bar{\Omega}}^{*} \circ \mathcal{P}_{\bar{T}}^{*}=\mathcal{P}_{\bar{\Omega}} \circ \mathcal{P}_{\bar{T}}$ because $\mathcal{P}_{\bar{\Omega}}$ and $\mathcal{P}_{\bar{T}}$ are self-adjoint, and similarly $\left(\mathcal{P}_{\bar{\Omega}} \circ \mathcal{P}_{\bar{T}}\right)^{*}=\mathcal{P}_{\bar{T}} \circ \mathcal{P}_{\bar{\Omega}}$. Now the claim follows by Proposition 3 and Lemma 9 using the facts that $\|\cdot\|_{b(\rho)}$ is dual to $\|\cdot\|_{\sharp(\rho)}$ and that $\|\cdot\|_{\text {vec }(1)}$ is dual to $\|\cdot\|_{\text {vec }(\infty)}$.

Note that Lemma 1 is encompassed by Lemma 10. Another consequence of these contraction properties is the following uncertainty principle, analogous to one stated by Chandrasekaran et al. (2009), which effectively states that a matrix $X$ cannot have both $\|\operatorname{sign}(X)\|_{\sharp(\rho)}$ and $\|\operatorname{orth}(X)\|_{\operatorname{vec}(\infty)}$ simultaneously small.
Theorem 4. If $X=\bar{X}_{S}=\bar{X}_{L} \neq 0$, then $\inf _{\rho>0} \alpha(\rho) \beta(\rho) \geq 1$.
Proof. Note that the non-zero element $X$ lives in $\bar{\Omega} \cap \bar{T}$, so we get the conclusion by the contrapositive of Theorem 1

### 4.2 Dual certificate

The incoherence properties allow us to construct an approximate dual certificate $\left(Q_{\bar{\Omega}}, Q_{\bar{T}}\right) \in \bar{\Omega} \times \bar{T}$ that is central to the analysis of the optimization problems (1) and (2).

The certificate is constructed as the solution to the linear system

$$
\left\{\begin{array}{llr}
\mathcal{P}_{\bar{\Omega}}\left(Q_{\bar{\Omega}}+Q_{\bar{T}}+\mu^{-1} E\right) & = & \lambda \operatorname{sign}\left(\bar{X}_{S}\right) \\
\mathcal{P}_{\bar{T}}\left(Q_{\bar{\Omega}}+Q_{\bar{T}}+\mu^{-1} E\right) & = & \operatorname{orth}\left(\bar{X}_{L}\right)
\end{array}\right.
$$

for some matrix $E \in \mathbb{R}^{m \times n}$; this can be equivalently written as

$$
\left[\begin{array}{cc}
\mathcal{I} & \mathcal{P}_{\bar{\Omega}} \\
\mathcal{P}_{\bar{T}} & \mathcal{I}
\end{array}\right]\left[\begin{array}{c}
Q_{\bar{\Omega}} \\
Q_{\bar{T}}
\end{array}\right]=\left[\begin{array}{c}
\lambda \operatorname{sign}\left(\bar{X}_{S}\right)-\mu^{-1} \mathcal{P}_{\bar{\Omega}}(E) \\
\operatorname{orth}\left(\bar{X}_{L}\right)-\mu^{-1} \mathcal{P}_{\bar{T}}(E)
\end{array}\right] .
$$

We show the existence of the dual certificate $\left(Q_{\bar{\Omega}}, Q_{\bar{T}}\right)$ under the conditions (3), (4), and (5) relative to an arbitrary matrix $E$. Recall that the recovery guarantees for the constrained formulation requires the conditions with $E=0$, while the guarantees for the regularized formulation takes $E=Y-\left(\bar{X}_{S}+\bar{X}_{L}\right)$.
Theorem 5. Pick any $c>1, \rho>0$, and $E \in \mathbb{R}^{m \times n}$. Let $\bar{k}:=\left|\operatorname{supp}\left(\bar{X}_{S}\right)\right|$ and $\bar{r}:=\operatorname{rank}\left(\bar{X}_{L}\right)$. Let

$$
\begin{aligned}
\epsilon_{2 \rightarrow 2} & :=\|E\|_{2 \rightarrow 2} \\
\epsilon_{\operatorname{vec}(\infty)} & :=\|E\|_{\operatorname{vec}(\infty)}+\left\|\mathcal{P}_{\bar{T}}(E)\right\|_{\operatorname{vec}(\infty)} .
\end{aligned}
$$

If the following conditions hold:

$$
\begin{align*}
& \alpha(\rho) \beta(\rho)<1  \tag{13}\\
& \lambda \leq \frac{(1-\alpha(\rho) \beta(\rho))\left(1-c \cdot \mu^{-1} \epsilon_{2 \rightarrow 2}\right)-c \cdot \alpha(\rho) \mu^{-1} \epsilon_{\mathrm{vec}(\infty)}-c \cdot \alpha(\rho) \gamma}{c \cdot \alpha(\rho)}  \tag{14}\\
& \lambda \geq c \cdot \frac{\gamma+\mu^{-1}(2-\alpha(\rho) \beta(\rho)) \epsilon_{\mathrm{vec}(\infty)}}{1-\alpha(\rho) \beta(\rho)-c \cdot \alpha(\rho) \beta(\rho)}>0 \tag{15}
\end{align*}
$$

(these are a restatement of (3), (4), and (5) ), then

$$
\begin{aligned}
& Q_{\bar{\Omega}}:=\left(\mathcal{I}-\mathcal{P}_{\bar{\Omega}} \circ \mathcal{P}_{\bar{T}}\right)^{-1}\left(\lambda \operatorname{sign}\left(\bar{X}_{S}\right)-\mathcal{P}_{\bar{\Omega}}\left(\operatorname{orth}\left(\bar{X}_{L}\right)\right)-\mu^{-1}\left(\mathcal{P}_{\bar{\Omega}} \circ \mathcal{P}_{\bar{T}^{\perp}}\right)(E)\right) \in \bar{\Omega} \quad \text { and } \\
& Q_{\bar{T}}:=\left(\mathcal{I}-\mathcal{P}_{\bar{T}} \circ \mathcal{P}_{\bar{\Omega}}\right)^{-1}\left(\operatorname{orth}\left(\bar{X}_{L}\right)-\lambda \mathcal{P}_{\bar{T}}\left(\operatorname{sign}\left(\bar{X}_{S}\right)\right)-\mu^{-1}\left(\mathcal{P}_{\bar{T}} \circ \mathcal{P}_{\bar{\Omega}^{\perp}}\right)(E)\right) \in \bar{T}
\end{aligned}
$$

are well-defined and satisfy

$$
\begin{aligned}
& \mathcal{P}_{\bar{\Omega}}\left(Q_{\bar{\Omega}}+Q_{\bar{T}}+\mu^{-1} E\right)=\lambda \operatorname{sign}\left(\bar{X}_{S}\right) \\
& \mathcal{P}_{\bar{T}}\left(Q_{\bar{\Omega}}+Q_{\bar{T}}+\mu^{-1} E\right)=\operatorname{orth}\left(\bar{X}_{L}\right)
\end{aligned}
$$

and

$$
\begin{array}{ll}
\left\|\mathcal{P}_{\bar{\Omega}^{\perp}}\left(Q_{\bar{\Omega}}+Q_{\bar{T}}+\mu^{-1} E\right)\right\|_{\operatorname{vec}(\infty)} & \leq \lambda / c \\
\left\|\mathcal{P}_{\bar{T}^{\perp}}\left(Q_{\bar{\Omega}}+Q_{\bar{T}}+\mu^{-1} E\right)\right\|_{2 \rightarrow 2} & \leq 1 / c .
\end{array}
$$

Moreover,

$$
\begin{aligned}
\left\|Q_{\bar{\Omega}}\right\|_{2 \rightarrow 2} & \leq \frac{\alpha(\rho)}{1-\alpha(\rho) \beta(\rho)} \cdot\left(\lambda+\gamma+\mu^{-1} \epsilon_{\operatorname{vec}(\infty)}\right) \\
\left\|Q_{\bar{T}}\right\|_{2 \rightarrow 2} & \leq \frac{2 \alpha(\rho)}{1-\alpha(\rho) \beta(\rho)} \cdot\left(\lambda+\gamma+\mu^{-1} \epsilon_{\operatorname{vec}(\infty)}\right)+1+2 \mu^{-1} \epsilon_{2 \rightarrow 2} \\
\left\|Q_{\bar{T}}\right\|_{*} & \leq 2 \bar{r}\left\|Q_{\bar{T}}\right\|_{2 \rightarrow 2} \\
\left\|Q_{\bar{T}}\right\|_{\operatorname{vec}(\infty)} & \leq \frac{1}{1-\alpha(\rho) \beta(\rho)} \cdot\left(\lambda+\gamma+\mu^{-1} \epsilon_{\operatorname{vec}(\infty)}\right) \\
\left\|Q_{\bar{\Omega}}\right\|_{\operatorname{vec}(\infty)} & \leq \frac{2}{1-\alpha(\rho) \beta(\rho)} \cdot\left(\lambda+\gamma+\mu^{-1} \epsilon_{\operatorname{vec}(\infty)}\right) \\
\left\|Q_{\bar{\Omega}}\right\|_{\operatorname{vec}(1)} & \leq \bar{k}\left\|Q_{\bar{\Omega}}\right\|_{\operatorname{vec}(\infty)} \\
\left\|Q_{\bar{\Omega}}+Q_{\bar{T}}\right\|_{\operatorname{vec}(2)}^{2} & \leq \lambda\left\|Q_{\bar{\Omega}}\right\|_{\operatorname{vec}(1)}\left(1+\mu^{-1} \lambda^{-1} \epsilon_{\operatorname{vec}(\infty)}\right)+\left\|Q_{\bar{T}}\right\|_{*}\left(1+2 \mu^{-1} \epsilon_{2 \rightarrow 2}\right)
\end{aligned}
$$

Remark 1. The dual certificate constitutes an approximate subgradient in the sense that $Q_{\bar{\Omega}}+Q_{\bar{T}}+\mu^{-1} E$ is a subgradient of both $\lambda\left\|X_{S}\right\|_{\mathrm{vec}(1)}$ at $X_{S}=\bar{X}_{S}$, and $\left\|X_{L}\right\|_{*}$ at $X_{L}=\bar{X}_{L}$.

Proof. Under the condition (13), we have $\alpha(\rho) \beta(\rho)<1$, and therefore Lemma 9 and Lemma 4 imply that the operators $\mathcal{I}-\mathcal{P}_{\bar{\Omega}} \circ \mathcal{P}_{\bar{T}}$ and $\mathcal{I}-\mathcal{P}_{\bar{T}} \circ \mathcal{P}_{\bar{\Omega}}$ are invertible and satisfy

$$
\max \left\{\left\|\left(\mathcal{I}-\mathcal{P}_{\bar{\Omega}} \circ \mathcal{P}_{\bar{T}}\right)^{-1}\right\|_{\sharp(\rho) \rightarrow \sharp(\rho)},\left\|\left(\mathcal{I}-\mathcal{P}_{\bar{T}} \circ \mathcal{P}_{\bar{\Omega}}\right)^{-1}\right\|_{\operatorname{vec}(\infty) \rightarrow \operatorname{vec}(\infty)}\right\} \leq \frac{1}{1-\alpha(\rho) \beta(\rho)}
$$

Thus $Q_{\bar{\Omega}}$ and $Q_{\bar{T}}$ are well-defined. We can bound $\left\|Q_{\bar{\Omega}}\right\|_{2 \rightarrow 2}$ as

$$
\begin{aligned}
\left\|Q_{\bar{\Omega}}\right\|_{2 \rightarrow 2} & \leq\left\|Q_{\bar{\Omega}}\right\|_{\sharp(\rho)} \quad(\operatorname{Lemman} 2) \\
& =\left\|\left(I-\mathcal{P}_{\bar{\Omega}} \circ \mathcal{P}_{\bar{T}}\right)^{-1}\left(\lambda \operatorname{sign}\left(\bar{X}_{S}\right)-\mathcal{P}_{\bar{\Omega}}\left(\operatorname{orth}\left(\bar{X}_{L}\right)\right)-\mu^{-1}\left(\mathcal{P}_{\bar{\Omega}} \circ \mathcal{P}_{\bar{T}^{\perp}}\right)(E)\right)\right\|_{\sharp(\rho)} \\
& \leq \frac{1}{1-\alpha(\rho) \beta(\rho)} \cdot\left\|\lambda \operatorname{sign}\left(\bar{X}_{S}\right)-\mathcal{P}_{\bar{\Omega}}\left(\operatorname{orth}\left(\bar{X}_{L}\right)\right)-\mu^{-1}\left(\mathcal{P}_{\bar{\Omega}} \circ \mathcal{P}_{\bar{T}^{\perp}}\right)(E)\right\|_{\sharp(\rho)} \\
& \leq \frac{1}{1-\alpha(\rho) \beta(\rho)} \cdot\left(\lambda\left\|\operatorname{sign}\left(\bar{X}_{S}\right)\right\|_{\sharp(\rho)}+\left\|\mathcal{P}_{\bar{\Omega}}\left(\operatorname{orth}\left(\bar{X}_{L}\right)\right)\right\|_{\sharp(\rho)}+\mu^{-1}\left\|\left(\mathcal{P}_{\bar{\Omega}^{\prime}} \circ \mathcal{P}_{\bar{T}^{\perp}}\right)(E)\right\|_{\sharp(\rho)}\right) \\
& \leq \frac{\alpha(\rho)}{1-\alpha(\rho) \beta(\rho)} \cdot\left(\lambda+\gamma+\mu^{-1}\left\|\mathcal{P}_{\bar{T}^{\perp}}(E)\right\|_{\operatorname{vec}(\infty)}\right) \quad(\text { Lemma 7 }) \\
& \leq \frac{\alpha(\rho)}{1-\alpha(\rho) \beta(\rho)} \cdot\left(\lambda+\gamma+\mu^{-1} \epsilon_{\operatorname{vec}(\infty)}\right) .
\end{aligned}
$$

Above, we have used the bound $\left\|\mathcal{P}_{\bar{T} \perp}(E)\right\|_{\mathrm{vec}(\infty)}=\left\|E-\mathcal{P}_{\bar{T}}(E)\right\|_{\mathrm{vec}(\infty)} \leq \epsilon_{\mathrm{vec}(\infty)}$. Therefore,

$$
\begin{aligned}
\left\|\mathcal{P}_{\bar{T}^{\perp}}\left(Q_{\bar{\Omega}}+\mu^{-1} E\right)\right\|_{2 \rightarrow 2} & \leq\left\|\left(I-\bar{U} \bar{U}^{\top}\right) Q_{\bar{\Omega}}\left(I-\bar{V} \bar{V}^{\top}\right)\right\|_{2 \rightarrow 2}+\mu^{-1}\left\|P_{\bar{T}^{\perp}}(E)\right\|_{2 \rightarrow 2} \\
& \leq\left\|Q_{\bar{\Omega}}\right\|_{2 \rightarrow 2}+\mu^{-1} \epsilon_{2 \rightarrow 2} \\
& \leq \frac{\alpha(\rho)}{1-\alpha(\rho) \beta(\rho)} \cdot\left(\lambda+\gamma+\mu^{-1} \epsilon_{\operatorname{vec}(\infty)}\right)+\mu^{-1} \epsilon_{2 \rightarrow 2} .
\end{aligned}
$$

The condition (14) now implies that this quantity is at most $1 / c$.
Now we bound $\left\|Q_{\bar{T}}\right\|_{\text {vec }(\infty)}$ as

$$
\begin{aligned}
\left\|Q_{\bar{T}}\right\|_{\operatorname{vec}(\infty)} & =\|\left(\mathcal{I}-\mathcal{P}_{\bar{T}} \circ \mathcal{P}_{\bar{\Omega}}\right)^{-1}\left(\operatorname{orth}\left(\bar{X}_{L}\right)-\lambda \mathcal{P}_{\bar{T}}\left(\operatorname{sign}\left(\bar{X}_{S}\right)-\mu^{-1}\left(\mathcal{P}_{\bar{T}^{\prime}} \circ \mathcal{P}_{\bar{\Omega}^{\perp}}\right)(E)\right) \|_{\operatorname{vec}(\infty)}\right. \\
& \leq \frac{1}{1-\alpha(\rho) \beta(\rho)} \cdot \| \operatorname{orth}\left(\bar{X}_{L}\right)-\lambda \mathcal{P}_{\bar{T}}\left(\operatorname{sign}\left(\bar{X}_{S}\right)-\mu^{-1}\left(\mathcal{P}_{\bar{T}} \circ \mathcal{P}_{\bar{\Omega}^{\perp}}\right)(E) \|_{\operatorname{vec}(\infty)}\right. \\
& \leq \frac{1}{1-\alpha(\rho) \beta(\rho)} \cdot\left(\left\|\operatorname{orth}\left(\bar{X}_{L}\right)\right\|_{\operatorname{vec}(\infty)}+\lambda\left\|\mathcal{P}_{\bar{T}}\left(\operatorname{sign}\left(\bar{X}_{S}\right)\right)\right\|_{\operatorname{vec}(\infty)}+\mu^{-1}\left\|\left(\mathcal{P}_{\bar{T}} \circ \mathcal{P}_{\bar{\Omega}^{\perp}}\right)(E)\right\|_{\operatorname{vec}(\infty)}\right) \\
& \leq \frac{1}{1-\alpha(\rho) \beta(\rho)} \cdot\left(\gamma+\lambda \alpha(\rho) \beta(\rho)+\mu^{-1} \epsilon_{\operatorname{vec}(\infty)}\right) \quad(\text { Lemma } 9) .
\end{aligned}
$$

Above, we have used the bound $\left\|\left(\mathcal{P}_{\bar{T}^{\circ}} \circ \mathcal{P}_{\bar{\Omega}^{\perp}}\right)(E)\right\|_{\mathrm{vec}(\infty)}=\left\|\mathcal{P}_{\bar{T}}(E)-\left(\mathcal{P}_{\bar{T}} \circ \mathcal{P}_{\bar{\Omega}}\right)(E)\right\|_{\mathrm{vec}(\infty)} \leq\left\|\mathcal{P}_{\bar{T}}(E)\right\|_{\operatorname{vec}(\infty)}+$ $\alpha(\rho) \beta(\rho)\|E\|_{\mathrm{vec}(\infty)} \leq \epsilon_{\mathrm{vec}(\infty)}$. Therefore,

$$
\begin{aligned}
\left\|\mathcal{P}_{\bar{\Omega}^{\perp}}\left(Q_{\bar{T}}+\mu^{-1} E\right)\right\|_{\operatorname{vec}(\infty)} & \leq\left\|Q_{\bar{T}}\right\|_{\operatorname{vec}(\infty)}+\mu^{-1}\left\|\mathcal{P}_{\bar{\Omega}^{\perp}}(E)\right\|_{\operatorname{vec}(\infty)} \\
& \leq \frac{1}{1-\alpha(\rho) \beta(\rho)} \cdot\left(\gamma+\lambda \alpha(\rho) \beta(\rho)+\mu^{-1} \epsilon_{\operatorname{vec}(\infty)}\right)+\mu^{-1} \epsilon_{\operatorname{vec}(\infty)} .
\end{aligned}
$$

The condition (15) now implies that this quantity is at most $\lambda / c$.
We also have

$$
\begin{aligned}
\left\|Q_{\bar{T}}\right\|_{2 \rightarrow 2} & =\left\|\mathcal{P}_{\bar{T}}\left(Q_{\bar{\Omega}}+\mu^{-1} E\right)-\operatorname{orth}\left(\bar{X}_{L}\right)\right\|_{2 \rightarrow 2} \\
& \leq \frac{2 \alpha(\rho)}{1-\alpha(\rho) \beta(\rho)} \cdot\left(\lambda+\gamma+\mu^{-1} \epsilon_{\operatorname{vec}(\infty)}\right)+1+2 \mu^{-1} \epsilon_{2 \rightarrow 2}
\end{aligned}
$$

since $\left\|\mathcal{P}_{\bar{T}}\left(Q_{\bar{\Omega}}\right)\right\|_{2 \rightarrow 2} \leq 2\left\|Q_{\bar{\Omega}}\right\|_{2 \rightarrow 2}$ and $\left\|\mathcal{P}_{\bar{T}}(E)\right\|_{2 \rightarrow 2} \leq 2 \epsilon_{2 \rightarrow 2}$ by Lemma 區 and

$$
\begin{aligned}
\left\|Q_{\bar{\Omega}}\right\|_{\operatorname{vec}(\infty)} & =\left\|P_{\bar{\Omega}}\left(Q_{\bar{T}}+\mu^{-1} E\right)-\lambda \operatorname{sign}\left(\bar{X}_{S}\right)\right\|_{\operatorname{vec}(\infty)} \\
& \leq \frac{1}{1-\alpha(\rho) \beta(\rho)} \cdot\left(\lambda+\gamma+\mu^{-1} \epsilon_{\operatorname{vec}(\infty)}\right)+\lambda+\mu^{-1} \epsilon_{\operatorname{vec}(\infty)} .
\end{aligned}
$$

The bounds on $\left\|Q_{\bar{T}}\right\|_{*}$ and $\left\|Q_{\bar{\Omega}}\right\|_{\operatorname{vec}(1)}$ follow from the facts that $\operatorname{rank}\left(Q_{\bar{T}}\right) \leq 2 \bar{r}$ and $\left\|\operatorname{supp}\left(Q_{\bar{\Omega}}\right)\right\| \leq \bar{k}$. Finally,

$$
\begin{aligned}
\left\|Q_{\bar{\Omega}}+Q_{\bar{T}}\right\|_{\operatorname{vec}(2)}^{2} & =\left\langle Q_{\bar{\Omega}}, \mathcal{P}_{\bar{\Omega}}\left(Q_{\bar{\Omega}}+Q_{\bar{T}}\right)\right\rangle+\left\langle Q_{\bar{T}}, \mathcal{P}_{\bar{T}}\left(Q_{\bar{\Omega}}+Q_{\bar{T}}\right)\right\rangle \\
& =\left\langle Q_{\bar{\Omega}}, \lambda \mathcal{P}_{\bar{\Omega}}\left(\operatorname{sign}\left(\bar{X}_{S}\right)\right)-\mu^{-1} \mathcal{P}_{\bar{\Omega}}(E)\right\rangle+\left\langle Q_{\bar{T}}, \mathcal{P}_{\bar{T}}\left(\operatorname{orth}\left(\bar{X}_{L}\right)\right)-\mu^{-1} \mathcal{P}_{\bar{T}}(E)\right\rangle \\
& \leq \lambda\left\|Q_{\bar{\Omega}}\right\|_{\operatorname{vec}(1)}\left(1+\mu^{-1} \lambda^{-1}\left\|\mathcal{P}_{\bar{\Omega}}(E)\right\|_{\operatorname{vec}(\infty)}\right)+\left\|Q_{\bar{T}}\right\|_{*}\left(1+\mu^{-1}\left\|\mathcal{P}_{\bar{T}}(E)\right\|_{2 \rightarrow 2}\right) \\
& \leq \lambda\left\|Q_{\bar{\Omega}}\right\|_{\operatorname{vec}(1)}\left(1+\mu^{-1} \lambda^{-1} \epsilon_{\operatorname{vec}(\infty)}\right)+\left\|Q_{\bar{T}}\right\|_{*}\left(1+2 \mu^{-1} \epsilon_{2 \rightarrow 2}\right) .
\end{aligned}
$$

## 5 Analysis of constrained formulation

Throughout this section, we fix a target decomposition $\left(\bar{X}_{S}, \bar{X}_{L}\right)$ that satisfies the constraints of (1), and let ( $\hat{X}_{S}, \hat{X}_{L}$ ) be the optimal solution to (11). Let $\Delta_{S}:=\hat{X}_{S}-\bar{X}_{S}$ and $\Delta_{L}:=\hat{X}_{L}-\bar{X}_{L}$. We show that under the
conditions of Theorem 5 with $E=0$ and appropriately chosen $\lambda$, solving (1) accurately recovers the target decomposition $\left(\bar{X}_{S}, \bar{X}_{L}\right)$.

We decompose the errors into symmetric and antisymmetric parts $\Delta_{\text {avg }}:=\left(\Delta_{S}+\Delta_{L}\right) / 2$ and $\Delta_{\text {mid }}:=$ $\left(\Delta_{S}-\Delta_{L}\right) / 2$. The constraints allow us to easily bound $\Delta_{\text {avg }}$, so most of the analysis involves bounding $\Delta_{\text {mid }}$ in terms of $\Delta_{\text {avg }}$.
Lemma 11. $\left\|\Delta_{\text {avg }}\right\|_{\operatorname{vec}(1)} \leq \epsilon_{\mathrm{vec}(1)}$ and $\left\|\Delta_{\mathrm{avg}}\right\|_{*} \leq \epsilon_{*}$.
Proof. Since both $\left(\hat{X}_{S}, \hat{X}_{L}\right)$ and $\left(\bar{X}_{S}, \bar{X}_{L}\right)$ as feasible solutions to (11), we have for $\diamond \in\{\operatorname{vec}(1), *\}$,

$$
\begin{aligned}
\left\|\Delta_{\text {avg }}\right\|_{\diamond} & =1 / 2\left\|\Delta_{S}+\Delta_{L}\right\|_{\diamond} \\
& =1 / 2\left\|\left(\hat{X}_{S}+\hat{X}_{L}-Y\right)-\left(\bar{X}_{S}+\bar{X}_{L}-Y\right)\right\|_{\diamond} \\
& \leq 1 / 2\left(\left\|\hat{X}_{S}+\hat{X}_{L}-Y\right\|_{\diamond}+\left\|\bar{X}_{S}+\bar{X}_{L}-Y\right\|_{\diamond}\right) \\
& \leq \epsilon_{\diamond} .
\end{aligned}
$$

Lemma 12. Assume the conditions of Theorem 5 hold with $E=0$. We have

$$
\lambda\left\|\mathcal{P}_{\bar{\Omega}^{\perp}}\left(\Delta_{\operatorname{mid}}\right)\right\|_{\operatorname{vec}(1)}+\left\|\mathcal{P}_{\bar{T}^{\perp}}\left(\Delta_{\operatorname{mid}}\right)\right\|_{*} \leq(1-1 / c)^{-1}\left(\lambda\left\|\Delta_{\text {avg }}\right\|_{\operatorname{vec}(1)}+\left\|\Delta_{\text {avg }}\right\|_{*}\right)
$$

Proof. Let $Q:=Q_{\bar{\Omega}}+Q_{\bar{T}}$ be the dual certificate guaranteed by Theorem 5. Note that $Q$ satisfies the conditions of Lemma 6, so we have

$$
\begin{aligned}
& \lambda\left\|\bar{X}_{S}+\Delta_{\mathrm{mid}}\right\|_{\operatorname{vec}(1)}+\left\|\bar{X}_{L}-\Delta_{\mathrm{mid}}\right\|_{*}-\lambda\left\|\bar{X}_{S}\right\|_{\operatorname{vec}(1)}-\left\|\bar{X}_{L}\right\|_{*} \\
& \geq(1-1 / c)\left(\lambda\left\|\mathcal{P}_{\bar{\Omega}^{\perp}}\left(\Delta_{\mathrm{mid}}\right)\right\|_{\operatorname{vec}(1)}+\left\|\mathcal{P}_{\bar{T}^{\perp}}\left(\Delta_{\mathrm{mid}}\right)\right\|_{*}\right) .
\end{aligned}
$$

Using the triangle inequality, we have

$$
\begin{aligned}
\lambda\left\|\hat{X}_{S}\right\|_{\operatorname{vec}(1)}+\left\|\hat{X}_{L}\right\|_{*} & =\lambda\left\|\bar{X}_{S}+\Delta_{S}\right\|_{\operatorname{vec}(1)}+\left\|\bar{X}_{L}+\Delta_{L}\right\|_{*} \\
& =\lambda\left\|\bar{X}_{S}+\Delta_{\operatorname{mid}}+\Delta_{\mathrm{avg}}\right\|_{\operatorname{vec}(1)}+\left\|\bar{X}_{L}-\Delta_{\mathrm{mid}}+\Delta_{\mathrm{avg}}\right\|_{*} \\
& \geq \lambda\left\|\bar{X}_{S}+\Delta_{\operatorname{mid}}\right\|_{\operatorname{vec}(1)}-\lambda\left\|\Delta_{\mathrm{avg}}\right\|_{\mathrm{vec}(1)}+\left\|\bar{X}_{L}-\Delta_{\mathrm{mid}}\right\|_{*}-\left\|\Delta_{\mathrm{avg}}\right\|_{*} .
\end{aligned}
$$

Now using the fact that $\lambda\left\|\hat{X}_{S}\right\|_{\text {vec(1) }}+\left\|\hat{X}_{L}\right\|_{*} \leq \lambda\left\|\bar{X}_{S}\right\|_{\operatorname{vec}(1)}+\left\|\bar{X}_{L}\right\|_{*}$ gives the claim.
Lemma 13. Let $\bar{k}:=\left|\operatorname{supp}\left(\bar{X}_{S}\right)\right|$. Assume the conditions of Theorem 5 hold with $E=0$. We have

$$
\left\|\mathcal{P}_{\bar{\Omega}}\left(\Delta_{\text {mid }}\right)\right\|_{\operatorname{vec}(1)} \leq \frac{(1-1 / c)^{-1}}{1-\alpha(\rho) \beta(\rho)} \cdot\left(\left\|\Delta_{\text {avg }}\right\|_{\operatorname{vec}(1)}+\left\|\Delta_{\text {avg }}\right\|_{*} / \lambda\right)
$$

Proof. Because $\Delta_{\text {mid }}=\mathcal{P}_{\bar{\Omega}}\left(\Delta_{\text {mid }}\right)+\mathcal{P}_{\bar{\Omega}^{\perp}}\left(\Delta_{\text {mid }}\right)=\mathcal{P}_{\bar{T}}\left(\Delta_{\text {mid }}\right)+\mathcal{P}_{\bar{T}^{\perp}}\left(\Delta_{\text {mid }}\right)$, we have the equation

$$
\mathcal{P}_{\bar{\Omega}}\left(\Delta_{\text {mid }}\right)-\mathcal{P}_{\bar{T}}\left(\Delta_{\text {mid }}\right)=-\mathcal{P}_{\bar{\Omega}^{\perp}}\left(\Delta_{\text {mid }}\right)+\mathcal{P}_{\bar{T}^{\perp}}\left(\Delta_{\text {mid }}\right)
$$

Separately applying $\mathcal{P}_{\bar{\Omega}}$ and $\mathcal{P}_{\bar{T}}$ to both sides gives

$$
\left[\begin{array}{cc}
\mathcal{I} & \mathcal{P}_{\bar{\Omega}} \\
\mathcal{P}_{\bar{T}} & \mathcal{I}
\end{array}\right]\left[\begin{array}{c}
\mathcal{P}_{\bar{\Omega}}\left(\Delta_{\text {mid }}\right) \\
-\mathcal{P}_{\bar{T}}\left(\Delta_{\text {mid }}\right)
\end{array}\right]=\left[\begin{array}{c}
\left(\mathcal{P}_{\bar{\Omega}} \circ \mathcal{P}_{\bar{T}^{\perp}}\right)\left(\Delta_{\text {mid }}\right) \\
-\left(\mathcal{P}_{\bar{T}} \circ \mathcal{P}_{\bar{\Omega}^{\perp}}\right)\left(\Delta_{\text {mid }}\right)
\end{array}\right]
$$

Under the condition $\alpha(\rho) \beta(\rho)<1$, Lemma 10 and Lemma 4 imply that

$$
\left\|\left(\mathcal{I}-\mathcal{P}_{\bar{\Omega}} \circ \mathcal{P}_{\bar{T}}\right)^{-1}\right\|_{\operatorname{vec}(1) \rightarrow \operatorname{vec}(1)} \leq \frac{1}{1-\alpha(\rho) \beta(\rho)}
$$

and that

$$
\mathcal{P}_{\bar{\Omega}}\left(\Delta_{\text {mid }}\right)=\left(\mathcal{I}-\mathcal{P}_{\bar{\Omega}} \circ \mathcal{P}_{\bar{T}}\right)^{-1}\left(\left(\mathcal{P}_{\bar{\Omega}} \circ \mathcal{P}_{\bar{T}^{\perp}}\right)\left(\Delta_{\text {mid }}\right)-\left(\mathcal{P}_{\bar{\Omega}} \circ \mathcal{P}_{\bar{T}} \circ \mathcal{P}_{\bar{\Omega}^{\perp}}\right)\left(\Delta_{\text {mid }}\right)\right)
$$

Therefore

$$
\begin{align*}
& \left\|\mathcal{P}_{\bar{\Omega}}\left(\Delta_{\text {mid }}\right)\right\|_{\operatorname{vec}(1)} \\
& \leq \frac{1}{1-\alpha(\rho) \beta(\rho)} \cdot\left(\left\|\left(\mathcal{P}_{\bar{\Omega}} \circ \mathcal{P}_{\bar{T}^{\perp}}\right)\left(\Delta_{\operatorname{mid}}\right)\right\|_{\operatorname{vec}(1)}+\left\|\left(\mathcal{P}_{\bar{\Omega}} \circ \mathcal{P}_{\bar{T}} \circ \mathcal{P}_{\bar{\Omega}^{\perp}}\right)\left(\Delta_{\operatorname{mid}}\right)\right\|_{\mathrm{vec}(1)}\right) \\
& \leq \frac{1}{1-\alpha(\rho) \beta(\rho)} \cdot\left(\sqrt{\bar{k}} \cdot\left\|\mathcal{P}_{\bar{T}^{\perp}}\left(\Delta_{\operatorname{mid}}\right)\right\|_{\operatorname{vec}(2)}+\alpha(\rho) \beta(\rho) \cdot\left\|\mathcal{P}_{\bar{\Omega}^{\perp}}\left(\Delta_{\operatorname{mid}}\right)\right\|_{\operatorname{vec}(1)}\right) \quad \text { (Lemma 10) }  \tag{Lemma10}\\
& \leq \frac{1}{1-\alpha(\rho) \beta(\rho)} \cdot\left(\sqrt{\bar{k}} \cdot\left\|\mathcal{P}_{\bar{T}^{\perp}}\left(\Delta_{\operatorname{mid}}\right)\right\|_{*}+\alpha(\rho) \beta(\rho) \cdot\left\|\mathcal{P}_{\bar{\Omega}^{\perp}}\left(\Delta_{\operatorname{mid}}\right)\right\|_{\operatorname{vec}(1)}\right) \\
& \leq \frac{(1-1 / c)^{-1}}{1-\alpha(\rho) \beta(\rho)} \cdot \max \{\sqrt{\bar{k}}, \alpha(\rho) \beta(\rho) / \lambda\} \cdot\left(\lambda\left\|\Delta_{\operatorname{avg}}\right\|_{\mathrm{vec}(1)}+\left\|\Delta_{\mathrm{avg}}\right\|_{*}\right) \quad \text { (Lemma 12) } \\
& \leq \frac{(1-1 / c)^{-1}}{1-\alpha(\rho) \beta(\rho)} \cdot\left(\left\|\Delta_{\operatorname{avg}}\right\|_{\operatorname{vec}(1)}+\left\|\Delta_{\operatorname{avg}}\right\|_{*} / \lambda\right)
\end{align*}
$$

where the last inequality uses the facts $\bar{k} \leq \alpha(\rho)^{2}, \alpha(\rho) \beta(\rho)<1$, and $\lambda \alpha(\rho) \leq 1$.
We now prove Theorem 2 which we restate here for convenience.
Theorem 6 (Theorem 2 restated). Assume the conditions of Theorem 5 hold with $E=0$. We have

$$
\begin{aligned}
& \max \left\{\left\|\Delta_{S}\right\|_{\operatorname{vec}(1)},\left\|\Delta_{L}\right\|_{\operatorname{vec}(1)}\right\} \\
& \quad \leq\left(1+(1-1 / c)^{-1} \cdot \frac{2-\alpha(\rho) \beta(\rho)}{1-\alpha(\rho) \beta(\rho)}\right) \cdot \epsilon_{\operatorname{vec}(1)}+(1-1 / c)^{-1} \cdot \frac{2-\alpha(\rho) \beta(\rho)}{1-\alpha(\rho) \beta(\rho)} \cdot \epsilon_{*} / \lambda
\end{aligned}
$$

If, in addition for some $b \geq\left\|\bar{X}_{L}\right\|_{\mathrm{vec}(\infty)}$, either:

- the optimization problem (1) is augmented with the constraint $\left\|X_{L}\right\|_{\operatorname{vec}(\infty)} \leq b$, or
- $\hat{X}_{L}$ is post-processed by replacing $\left[\hat{X}_{L}\right]_{i, j}$ with $\min \left\{\max \left\{\left[\hat{X}_{L}\right]_{i, j},-b\right\}, b\right\}$ for all $i, j$,
then we also have

$$
\left\|\Delta_{L}\right\|_{\mathrm{vec}(2)} \leq \min \left\{\left\|\Delta_{L}\right\|_{\mathrm{vec}(1)}, \sqrt{2 b\left\|\Delta_{L}\right\|_{\mathrm{vec}(1)}}\right\}
$$

Proof. First note that since $\Delta_{S}=\Delta_{\text {avg }}+\Delta_{\text {mid }}$ and $\Delta_{L}=\Delta_{\text {avg }}-\Delta_{\text {mid }}$, we have max $\left\{\left\|\Delta_{S}\right\|_{\text {vec }(1)},\left\|\Delta_{L}\right\|_{\text {vec }(1)}\right\} \leq$ $\left\|\Delta_{\text {avg }}\right\|_{\text {vec }(1)}+\left\|\Delta_{\text {mid }}\right\|_{\text {vec }(1)}$. We can bound $\left\|\Delta_{\text {mid }}\right\|_{\text {vec }(1)}$ as

$$
\begin{aligned}
\left\|\Delta_{\operatorname{mid}}\right\|_{\operatorname{vec}(1)} & \leq\left\|\mathcal{P}_{\bar{\Omega}^{\perp}}\left(\Delta_{\operatorname{mid}}\right)\right\|_{\operatorname{vec}(1)}+\left\|\mathcal{P}_{\bar{\Omega}}\left(\Delta_{\operatorname{mid}}\right)\right\|_{\operatorname{vec}(1)} \\
& \leq(1-1 / c)^{-1} \cdot\left(1+\frac{1}{1-\alpha(\rho) \beta(\rho)}\right) \cdot\left(\left\|\Delta_{\mathrm{avg}}\right\|_{\operatorname{vec}(1)}+\left\|\Delta_{\mathrm{avg}}\right\|_{*} / \lambda\right)
\end{aligned}
$$

by Lemma 12 and Lemma 13. The bounds on $\left\|\Delta_{S}\right\|_{\text {vec(1) }}$ and $\left\|\Delta_{L}\right\|_{\text {vec(1) }}$ follow from the bounds on $\left\|\Delta_{\text {mid }}\right\|_{\text {vec }(1)},\left\|\Delta_{\text {avg }}\right\|_{\text {vec }(1)}$, and $\left\|\Delta_{\text {avg }}\right\|_{*}$ (from Lemma 11).

If the constraint $\left\|X_{L}\right\|_{\mathrm{vec}(\infty)} \leq b$ is added, then we can use the facts $\left\|\Delta_{L}\right\|_{\mathrm{vec}(\infty)} \leq\left\|\hat{X}_{L}\right\|_{\mathrm{vec}(\infty)}+$ $\left\|\bar{X}_{L}\right\|_{\operatorname{vec}(\infty)} \leq 2 b$ and $\left\|\Delta_{L}\right\|_{\operatorname{vec}(2)} \sqrt{\left\|\Delta_{L}\right\|_{\operatorname{vec}(\infty)}\left\|\Delta_{L}\right\|_{\operatorname{vec}(1)}} \leq \sqrt{2 b\left\|\Delta_{L}\right\|_{\operatorname{vec}(1)}}$. If $\hat{X}_{L}$ is post-processed, then (letting $\operatorname{clip}\left(\hat{X}_{L}\right)$ be the result of the post-processing) $\left|\operatorname{clip}\left(\hat{X}_{L}\right)_{i, j}-\left[\bar{X}_{L}\right]_{i, j}\right| \leq\left|\left[\hat{X}_{L}\right]_{i, j}-\left[\bar{X}_{L}\right]_{i, j}\right|$ for all $i, j$, so $\left\|\operatorname{clip}\left(\hat{X}_{L}\right)-\bar{X}_{L}\right\|_{\operatorname{vec}(1)} \leq\left\|\Delta_{L}\right\|_{\operatorname{vec}(1)}$ and $\left\|\operatorname{clip}\left(\hat{X}_{L}\right)-\bar{X}_{L}\right\|_{\operatorname{vec}(2)} \leq \sqrt{2 b\left\|\operatorname{clip}\left(\hat{X}_{L}\right)-\bar{X}_{L}\right\|_{\operatorname{vec}(1)}}$.

## 6 Analysis of regularized formulation

Throughout this section, we fix a target decomposition $\left(\bar{X}_{S}, \bar{X}_{L}\right)$ that satisfies $\left\|\bar{X}_{S}-Y\right\|_{\mathrm{vec}(\infty)} \leq b$, and let $\left(\hat{X}_{S}, \hat{X}_{L}\right)$ be the optimal solution to (2) augmented with the constraint $\left\|X_{S}-Y\right\|_{\text {vec }(\infty)} \leq b$ for some $b \geq\left\|\bar{X}_{S}-Y\right\|_{\operatorname{vec}(\infty)}\left(b=\infty\right.$ is allowed). Let $\Delta_{S}:=\hat{X}_{S}-\bar{X}_{S}$ and $\Delta_{L}:=\hat{X}_{L}-\bar{X}_{L}$. We show that under the conditions of Theorem 5 with $E=Y-\left(\bar{X}_{S}+\bar{X}_{L}\right)$ and appropriately chosen $\lambda$ and $\mu$, solving (2) accurately recovers the target decomposition $\left(\bar{X}_{S}, \bar{X}_{L}\right)$.
Lemma 14. There exists $G_{S}, G_{L}, H_{S} \in \mathbb{R}^{m \times n}$ such that

1. $\mu^{-1}\left(\hat{X}_{S}+\hat{X}_{L}-Y\right)+\lambda G_{S}+H=0 ;\left\|G_{S}\right\|_{\mathrm{vec}(\infty)} \leq 1$;
2. $\mu^{-1}\left(\hat{X}_{S}+\hat{X}_{L}-Y\right)+\lambda G_{L}=0 ;\left\|G_{L}\right\|_{2 \rightarrow 2} \leq 1$;
3. $\left[H_{S}\right]_{i, j}\left[\Delta_{S}\right]_{i, j} \geq 0 \forall i, j$.

Proof. We express the constraint $\left\|X_{S}-Y\right\|_{\text {vec }(\infty)} \leq b$ in (22) as $2 m n$ constraints $\left[X_{S}\right]_{i, j}-Y_{i, j}-b \leq 0$ and $-\left[X_{S}\right]_{i, j}+Y_{i, j}-b \leq 0$ for all $i, j$. Now the corresponding Lagrangian is

$$
\frac{1}{2 \mu}\left\|X_{S}+X_{L}-Y\right\|_{\operatorname{vec}(2)}^{2}+\lambda\left\|X_{S}\right\|_{\operatorname{vec}(1)}+\left\|X_{L}\right\|_{*}+\left\langle\Lambda^{+}, X_{S}-Y-b 1_{m, n}\right\rangle+\left\langle\Lambda^{-},-X_{S}+Y-b 1_{m, n}\right\rangle
$$

where $\Lambda^{+}, \Lambda^{-} \geq 0$ and $1_{m, n}$ is the all-ones $m \times n$ matrix. First-order optimality conditions imply that there exists a subgradient $G_{S}$ of $\left\|X_{S}\right\|_{\text {vec(1) }}$ at $X_{S}=\hat{X}_{S}$ and a subgradient $G_{L}$ of $\left\|X_{L}\right\|_{*}$ at $X_{L}=\hat{X}_{L}$ such that

$$
\mu^{-1}\left(\hat{X}_{S}+\hat{X}_{L}-Y\right)+\lambda G_{S}+\left(\Lambda^{+}-\Lambda^{-}\right)=0 \quad \text { and } \quad \mu^{-1}\left(\hat{X}_{S}+\hat{X}_{L}-Y\right)+G_{L}=0
$$

Now since $\left\|\bar{X}_{S}-Y\right\|_{\operatorname{vec}(\infty)} \leq b$, we have $\left[\bar{X}_{S}\right]_{i, j} \leq Y_{i, j}+b$ and $-\left[\bar{X}_{S}\right]_{i, j} \leq-Y_{i, j}+b$. By complementary slackness, if $\Lambda_{i, j}^{+}>0$, then $\left[\hat{X}_{S}\right]_{i, j}-Y_{i, j}-b=0$, which means $\left[\hat{X}_{S}\right]_{i, j}-\left[\bar{X}_{S}\right]_{i, j} \geq\left[\hat{X}_{S}\right]_{i, j}-\left(Y_{i, j}+b\right)=0$. So $\Lambda_{i, j}^{+}\left[\Delta_{S}\right]_{i, j} \geq 0$. Similarly, if $\Lambda_{i, j}^{-}>0$, then $\left[\hat{X}_{S}\right]_{i, j}-\left[\bar{X}_{S}\right]_{i, j} \leq 0$. So $\Lambda_{i, j}^{-}\left[\Delta_{S}\right]_{i, j} \leq 0$. Therefore $H:=\Lambda^{+}-\Lambda^{-}$ satisfies $H_{i, j}\left[\Delta_{S}\right]_{i, j} \geq 0$.

Lemma 15. Assume the conditions of Theorem 5 hold with $E=Y-\left(\bar{X}_{S}+\bar{X}_{L}\right)$, and let $\left(Q_{\bar{\Omega}}, Q_{\bar{T}}\right)$ be the dual certificate from the conclusion. We have

$$
\lambda\left\|\mathcal{P}_{\bar{\Omega}^{\perp}}\left(\Delta_{S}\right)\right\|_{\operatorname{vec}(1)}+\left\|\mathcal{P}_{\bar{T}^{\perp}}\left(\Delta_{L}\right)\right\|_{*} \leq(1-1 / c)^{-1}\left\|Q_{\bar{\Omega}}+Q_{\bar{T}}\right\|_{\operatorname{vec}(2)}^{2} \mu / 2
$$

Proof. Let $Q:=Q_{\bar{\Omega}}+Q_{\bar{T}}$ and $\Delta:=\Delta_{S}+\Delta_{L}$. Since $Q+\mu^{-1} E$ satisfies the conditions of Lemma 6.

$$
\begin{aligned}
& (1-1 / c)\left(\lambda\left\|\mathcal{P}_{\bar{\Omega}^{\perp}}\left(\Delta_{S}\right)\right\|_{\operatorname{vec}(1)}+\left\|\mathcal{P}_{\bar{T}^{\perp}}\left(\Delta_{L}\right)\right\|_{*}\right) \\
& \quad \leq\left(\lambda\left\|\hat{X}_{S}\right\|_{\operatorname{vec}(1)}+\left\|\hat{X}_{L}\right\|_{*}\right)-\left(\lambda\left\|\bar{X}_{S}\right\|_{\operatorname{vec}(1)}+\left\|\bar{X}_{L}\right\|_{*}\right)-\left\langle Q+\mu^{-1} E, \Delta_{S}+\Delta_{L}\right\rangle
\end{aligned}
$$

Furthermore, by the optimality of $\left(\hat{X}_{S}, \hat{X}_{L}\right)$,

$$
\begin{aligned}
\left(\lambda\left\|\hat{X}_{S}\right\|_{\operatorname{vec}(1)}+\left\|\hat{X}_{L}\right\|_{*}\right)-\left(\lambda\left\|\bar{X}_{S}\right\|_{\operatorname{vec}(1)}+\left\|\bar{X}_{L}\right\|_{*}\right) & \leq \frac{1}{2 \mu}\left\|\bar{X}_{S}+\bar{X}_{L}-Y\right\|_{\operatorname{vec}(2)}^{2}-\frac{1}{2 \mu}\left\|\hat{X}_{S}+\hat{X}_{L}-Y\right\|_{\operatorname{vec}(2)}^{2} \\
& =\frac{1}{2 \mu}\|E\|_{\operatorname{vec}(2)}^{2}-\frac{1}{2 \mu}\left\|\Delta_{S}+\Delta_{L}-E\right\|_{\operatorname{vec}(2)}^{2} \\
& =\frac{1}{2 \mu}(2\langle E, \Delta\rangle-\langle\Delta, \Delta\rangle)
\end{aligned}
$$

Combining the inequalities gives

$$
(1-1 / c)\left(\lambda\left\|\mathcal{P}_{\bar{\Omega}^{\perp}}\left(\Delta_{S}\right)\right\|_{\operatorname{vec}(1)}+\left\|\mathcal{P}_{\bar{T}^{\perp}}\left(\Delta_{L}\right)\right\|_{*}\right) \leq-\langle Q, \Delta\rangle-\frac{1}{2 \mu}\langle\Delta, \Delta\rangle \leq\|Q\|_{\operatorname{vec}(2)}^{2} \mu / 2
$$

where the last inequality follows by taking the maximum value over $\Delta$ at $\Delta=-\mu Q$.

Now we prove Theorem 3, restated below (with an additional result for $\left\|\Delta_{L}\right\|_{b(\rho)}$ ).
Theorem 7 (Theorem 3 restated). Let $\bar{k}:=\left|\operatorname{supp}\left(\bar{X}_{S}\right)\right|$ and $\bar{r}:=\operatorname{rank}\left(\bar{X}_{L}\right)$. Assume the conditions of Theorem 5 hold with $E=Y-\left(\bar{X}_{S}+\bar{X}_{L}\right)$, and let $\left(Q_{\bar{\Omega}}, Q_{\bar{T}}\right)$ be the dual certificate from the conclusion. We have

$$
\begin{aligned}
\left\|\Delta_{S}\right\|_{\operatorname{vec}(1)} & \leq \frac{\lambda^{-1}(1-1 / c)^{-1}\left\|Q_{\bar{\Omega}}+Q_{\bar{T}}\right\|_{\operatorname{vec}(2)}^{2} \mu+\lambda \bar{k} \mu+2 \sqrt{\bar{k} \bar{r}} \mu+\bar{k}\left\|\left(\mathcal{P}_{\bar{\Omega}} \circ \mathcal{P}_{\bar{T}^{\perp}}\right)(E)\right\|_{\operatorname{vec}(\infty)}}{1-\alpha(\rho) \beta(\rho)} \\
\left\|\Delta_{S}\right\|_{\operatorname{vec}(2)} & \leq \min \left\{\left\|\Delta_{S}\right\|_{\operatorname{vec}(1)}, \sqrt{2 b\left\|\Delta_{S}\right\|_{\operatorname{vec}(1)}}\right\} \\
\left\|\Delta_{L}\right\|_{b(\rho)} & \leq(1-1 / c)^{-1}\left\|Q_{\bar{\Omega}}+Q_{\bar{T}}\right\|_{\operatorname{vec}(2)}^{2} \mu / 2+\min \left\{\beta(\rho)\left\|\Delta_{S}\right\|_{\operatorname{vec}(1)}, \sqrt{2 \bar{r}}\left\|\Delta_{S}\right\|_{\operatorname{vec}(2)}\right\}+\left\|\mathcal{P}_{\bar{T}}(E)\right\|_{*}+2 \bar{r} \mu \\
\left\|\Delta_{L}\right\|_{*} & \leq(1-1 / c)^{-1}\left\|Q_{\bar{\Omega}}+Q_{\bar{T}}\right\|_{\operatorname{vec}(2)}^{2} \mu / 2+\sqrt{2 \bar{r}}\left\|\Delta_{S}\right\|_{\operatorname{vec}(2)}+\left\|\mathcal{P}_{\bar{T}}(E)\right\|_{*}+2 \bar{r} \mu .
\end{aligned}
$$

Proof. From Lemma 14 we obtain $G_{S}, G_{L}, H_{S} \in \mathbb{R}^{m \times n}$ and the following equations:

$$
\begin{align*}
& \mu^{-1}\left(\mathcal{P}_{\bar{\Omega}}\left(\Delta_{S}\right)+\mathcal{P}_{\bar{\Omega}}\left(\Delta_{L}\right)-\mathcal{P}_{\bar{\Omega}}(E)\right)+\mathcal{P}_{\bar{\Omega}}\left(H_{S}\right)=-\lambda \mathcal{P}_{\bar{\Omega}}\left(G_{S}\right)  \tag{16}\\
& \mu^{-1}\left(\mathcal{P}_{\bar{T}}\left(\Delta_{S}\right)+\mathcal{P}_{\bar{T}}\left(\Delta_{L}\right)-\mathcal{P}_{\bar{T}}(E)\right)=-\mathcal{P}_{\bar{T}}\left(G_{L}\right)  \tag{17}\\
& \mu^{-1}\left(\left(\mathcal{P}_{\bar{\Omega}} \circ \mathcal{P}_{\bar{T}}\right)\left(\Delta_{S}\right)+\left(\mathcal{P}_{\bar{\Omega}} \circ \mathcal{P}_{\bar{T}}\right)\left(\Delta_{L}\right)-\left(\mathcal{P}_{\bar{\Omega}} \circ \mathcal{P}_{\bar{T}}\right)(E)\right)=-\left(\mathcal{P}_{\bar{\Omega}} \circ \mathcal{P}_{\bar{T}}\right)\left(G_{L}\right) \tag{18}
\end{align*}
$$

Subtracting (18) from (16) gives

$$
\begin{aligned}
& \mu^{-1}\left(\mathcal{P}_{\bar{\Omega}}\left(\Delta_{S}\right)-\left(\mathcal{P}_{\bar{\Omega}} \circ \mathcal{P}_{\bar{T}} \circ \mathcal{P}_{\bar{\Omega}}\right)\left(\Delta_{S}\right)-\left(\mathcal{P}_{\bar{\Omega}^{\circ}} \circ \mathcal{P}_{\bar{T}} \circ \mathcal{P}_{\bar{\Omega}^{\perp}}\right)\left(\Delta_{S}\right)-\left(\mathcal{P}_{\bar{\Omega}^{\circ}} \circ \mathcal{P}_{\bar{T}^{\perp}}\right)\left(\Delta_{L}\right)\right)+\mathcal{P}_{\bar{\Omega}}\left(H_{S}\right) \\
&=-\lambda \mathcal{P}_{\bar{\Omega}}\left(G_{S}\right)+\left(\mathcal{P}_{\bar{\Omega}^{\prime}} \circ \mathcal{P}_{\bar{T}}\right)\left(G_{L}\right)+\mu^{-1}\left(\mathcal{P}_{\bar{\Omega}^{\prime}} \circ \mathcal{P}_{\bar{T}^{\perp}}\right)(E)
\end{aligned}
$$

Moreover, we have $\left\langle\operatorname{sign}\left(\Delta_{S}\right), \mathcal{P}_{\bar{\Omega}}\left(\Delta_{S}\right)\right\rangle=\left\|\mathcal{P}_{\bar{\Omega}}\left(\Delta_{S}\right)\right\|_{\text {vec }(1)}$ and $\left\langle\operatorname{sign}\left(\Delta_{S}\right), \mathcal{P}_{\bar{\Omega}}\left(H_{S}\right)\right\rangle=\left\|\mathcal{P}_{\bar{\Omega}}\left(H_{S}\right)\right\|_{\operatorname{vec}(1)}$, so taking inner products with $\operatorname{sign}\left(\Delta_{S}\right)$ on both sides of the equation gives

$$
\begin{aligned}
& \mu^{-1}\left\|\mathcal{P}_{\bar{\Omega}}\left(\Delta_{S}\right)\right\|_{\operatorname{vec}(1)}+\left\|\mathcal{P}_{\bar{\Omega}}\left(H_{S}\right)\right\|_{\operatorname{vec}(1)} \\
& \leq \mu^{-1}\left\|\left(\mathcal{P}_{\bar{\Omega}} \circ \mathcal{P}_{\bar{T}} \circ \mathcal{P}_{\bar{\Omega}}\right)\left(\Delta_{S}\right)\right\|_{\operatorname{vec}(1)}+\mu^{-1}\left\|\left(\mathcal{P}_{\bar{\Omega}} \circ \mathcal{P}_{\bar{T}} \circ \mathcal{P}_{\bar{\Omega}^{\perp}}\right)\left(\Delta_{S}\right)\right\|_{\operatorname{vec}(1)}+\mu^{-1}\left\|\left(\mathcal{P}_{\bar{\Omega}} \circ \mathcal{P}_{\bar{T}^{\perp}}\right)\left(\Delta_{L}\right)\right\|_{\operatorname{vec}(1)} \\
& \quad+\lambda\left\|\mathcal{P}_{\bar{\Omega}}\left(G_{S}\right)\right\|_{\operatorname{vec}(1)}+\left\|\left(\mathcal{P}_{\bar{\Omega}} \circ \mathcal{P}_{\bar{T}}\right)\left(G_{L}\right)\right\|_{\operatorname{vec}(1)}+\mu^{-1}\left\|\left(\mathcal{P}_{\bar{\Omega}} \circ \mathcal{P}_{\bar{T}^{\perp}}\right)(E)\right\|_{\operatorname{vec}(1)} \\
& \leq \mu^{-1} \alpha(\rho) \beta(\rho)\left\|\mathcal{P}_{\bar{\Omega}}\left(\Delta_{S}\right)\right\|_{\operatorname{vec}(1)}+\mu^{-1} \alpha(\rho) \beta(\rho)\left\|\mathcal{P}_{\bar{\Omega}^{\perp}}\left(\Delta_{S}\right)\right\|_{\operatorname{vec}(1)}+\mu^{-1} \sqrt{\bar{k}}\left\|\mathcal{P}_{\bar{T}^{\perp}}\left(\Delta_{L}\right)\right\|_{\operatorname{vec}(2)} \\
& \quad+\lambda \bar{k}+\sqrt{\bar{k}}\left\|\mathcal{P}_{\bar{T}}\left(G_{L}\right)\right\|_{\operatorname{vec}(2)}+\mu^{-1} \bar{k}\left\|\left(\mathcal{P}_{\bar{\Omega}} \circ \mathcal{P}_{\bar{T}^{\perp}}\right)(E)\right\|_{\operatorname{vec}(\infty)} \\
& \leq \mu^{-1} \alpha(\rho) \beta(\rho)\left\|\mathcal{P}_{\bar{\Omega}}\left(\Delta_{S}\right)\right\|_{\operatorname{vec}(1)}+\mu^{-1} \alpha(\rho) \beta(\rho)\left\|\mathcal{P}_{\bar{\Omega}^{\perp}}\left(\Delta_{S}\right)\right\|_{\operatorname{vec}(1)}+\mu^{-1} \sqrt{\bar{k}}\left\|\mathcal{P}_{\bar{T}^{\perp}}\left(\Delta_{L}\right)\right\|_{\operatorname{vec}(2)} \\
& \quad+\lambda \bar{k}+2 \sqrt{\bar{k} \bar{r}}\left\|G_{L}\right\|_{2 \rightarrow 2}+\mu^{-1} \bar{k}\left\|\left(\mathcal{P}_{\bar{\Omega}} \circ \mathcal{P}_{\bar{T}^{\perp}}\right)(E)\right\|_{\operatorname{vec}(\infty)} \\
& \leq \mu^{-1} \alpha(\rho) \beta(\rho)\left\|\mathcal{P}_{\bar{\Omega}}\left(\Delta_{S}\right)\right\|_{\operatorname{vec}(1)}+\mu^{-1} \alpha(\rho) \beta(\rho)\left\|\mathcal{P}_{\bar{\Omega}^{\perp}}\left(\Delta_{S}\right)\right\|_{\operatorname{vec}(1)}+\mu^{-1} \sqrt{\bar{k}}\left\|\mathcal{P}_{\bar{T}^{\perp}}\left(\Delta_{L}\right)\right\|_{\operatorname{vec}(2)} \\
& \quad+\lambda \bar{k}+2 \sqrt{\bar{k} \bar{r}}+\mu^{-1} \bar{k}\left\|\left(\mathcal{P}_{\bar{\Omega}^{\prime}} \circ \mathcal{P}_{\bar{T}^{\perp}}\right)(E)\right\|_{\operatorname{vec}(\infty)} .
\end{aligned}
$$

The second and third inequalities above follow from Lemma 5 and Lemma 10, and the fourth inequality uses the fact that $\left\|G_{L}\right\|_{2 \rightarrow 2} \leq 1$. Rearranging the inequality and applying Lemma 15 gives

$$
\begin{aligned}
& (1-\alpha(\rho) \beta(\rho))\left\|\mathcal{P}_{\bar{\Omega}}\left(\Delta_{S}\right)\right\|_{\operatorname{vec}(1)} \\
& \leq \alpha(\rho) \beta(\rho)\left\|\mathcal{P}_{\bar{\Omega}^{\perp}}\left(\Delta_{S}\right)\right\|_{\operatorname{vec}(1)}+\sqrt{\bar{k}}\left\|\mathcal{P}_{\bar{T}^{\perp}}\left(\Delta_{L}\right)\right\|_{\operatorname{vec}(2)}+\lambda \bar{k} \mu+2 \sqrt{\bar{k} \bar{r}} \mu+\bar{k}\left\|\left(\mathcal{P}_{\bar{\Omega}} \circ \mathcal{P}_{\bar{T}^{\perp}}\right)(E)\right\|_{\operatorname{vec}(\infty)} \\
& \leq \max \{\alpha(\rho) \beta(\rho) / \lambda, \sqrt{\bar{k}}\}(1-1 / c)^{-1}\left\|Q_{\bar{\Omega}}+Q_{\bar{T}}\right\|_{\operatorname{vec}(2)}^{2} \mu / 2+\lambda \bar{k} \mu+2 \sqrt{\bar{k} \bar{r}} \mu+\bar{k}\left\|\left(\mathcal{P}_{\bar{\Omega}} \circ \mathcal{P}_{\bar{T}^{\perp}}\right)(E)\right\|_{\operatorname{vec}(\infty)} \\
& \leq \lambda^{-1}(1-1 / c)^{-1}\left\|Q_{\bar{\Omega}}+Q_{\bar{T}}\right\|_{\operatorname{vec}(2)}^{2} \mu / 2+\lambda \bar{k} \mu+2 \sqrt{\bar{k} \bar{r}} \mu+\bar{k}\left\|\left(\mathcal{P}_{\bar{\Omega}} \circ \mathcal{P}_{\bar{T}^{\perp}}\right)(E)\right\|_{\operatorname{vec}(\infty)}
\end{aligned}
$$

since $\bar{k} \leq \alpha(\rho)^{2}, \alpha(\rho) \beta(\rho)<1$, and $\lambda \alpha(\rho) \leq 1$. Now we combine this with $\left\|\Delta_{S}\right\|_{\operatorname{vec}(1)} \leq\left\|\mathcal{P}_{\bar{\Omega}^{\perp}}\left(\Delta_{S}\right)\right\|_{\operatorname{vec}(1)}+$ $\left\|\mathcal{P}_{\bar{\Omega}}\left(\Delta_{S}\right)\right\|_{\mathrm{vec}(1)}$ and Lemma 15 to get the first bound.

For the second bound, we use the facts $\left\|\Delta_{S}\right\|_{\operatorname{vec}(\infty)} \leq\left\|\hat{X}_{S}-Y\right\|_{\operatorname{vec}(\infty)}+\left\|\bar{X}_{S}-Y\right\|_{\operatorname{vec}(\infty)} \leq 2 b$ and $\left\|\Delta_{S}\right\|_{\text {vec }(2)} \leq \sqrt{\left\|\Delta_{S}\right\|_{\mathrm{vec}(1)}\left\|\Delta_{S}\right\|_{\mathrm{vec}(\infty)}} \leq \sqrt{2 b\left\|\Delta_{S}\right\|_{\mathrm{vec}(1)}}$.

For the third and fourth bounds, we obtain from (17)

$$
\begin{aligned}
\left\|\mathcal{P}_{\bar{T}}\left(\Delta_{L}\right)\right\|_{b(\rho)} & \leq\left\|\mathcal{P}_{\bar{T}}\left(\Delta_{S}\right)\right\|_{b(\rho)}+\left\|\mathcal{P}_{\bar{T}}(E)\right\|_{b(\rho)}+\mu\left\|\mathcal{P}_{\bar{T}}\left(G_{L}\right)\right\|_{b(\rho)} \\
& \leq\left\|\mathcal{P}_{\bar{T}}\right\|_{\operatorname{vec}(1) \rightarrow b(\rho)}\left\|\Delta_{S}\right\|_{\operatorname{vec}(1)}+\left\|\mathcal{P}_{\bar{T}}(E)\right\|_{*}+\mu\left\|\mathcal{P}_{\bar{T}}\left(G_{L}\right)\right\|_{*} \quad \text { (Lemma 3) } \\
& =\left\|\mathcal{P}_{\bar{T}}^{*}\right\|_{\sharp(\rho) \rightarrow \operatorname{vec}(\infty)}\left\|\Delta_{S}\right\|_{\operatorname{vec}(1)}+\left\|\mathcal{P}_{\bar{T}}(E)\right\|_{*}+\mu\left\|\mathcal{P}_{\bar{T}}\left(G_{L}\right)\right\|_{*} \quad \text { (Proposition 3) } \\
& \leq \beta(\rho)\left\|\Delta_{S}\right\|_{\operatorname{vec}(1)}+\left\|\mathcal{P}_{\bar{T}}(E)\right\|_{*}+\mu\left\|\mathcal{P}_{\bar{T}}\left(G_{L}\right)\right\|_{*} \quad \text { (Lemma 8 } \\
& \leq \beta(\rho)\left\|\Delta_{S}\right\|_{\operatorname{vec}(1)}+\left\|\mathcal{P}_{\bar{T}}(E)\right\|_{*}+2 \bar{r} \mu \quad \text { (Lemma 5 and }\left\|G_{L}\right\|_{2 \rightarrow 2} \leq 1 \text { ) }
\end{aligned}
$$

and

$$
\begin{aligned}
\left\|\mathcal{P}_{\bar{T}}\left(\Delta_{L}\right)\right\|_{*} & \leq\left\|\mathcal{P}_{\bar{T}}\left(\Delta_{S}\right)\right\|_{*}+\left\|\mathcal{P}_{\bar{T}}(E)\right\|_{*}+\mu\left\|\mathcal{P}_{\bar{T}}\left(G_{L}\right)\right\|_{*} \\
& \left.\leq \sqrt{2 \bar{r}}\left\|\Delta_{S}\right\|_{\operatorname{vec}(2)}+\left\|\mathcal{P}_{\bar{T}}(E)\right\|_{*}+2 \bar{r} \mu \quad \text { Lemma } 5 \text { and }\left\|G_{L}\right\|_{2 \rightarrow 2} \leq 1\right)
\end{aligned}
$$

Now we combine these with

$$
\begin{aligned}
\left\|\Delta_{L}\right\|_{b(\rho)} & \leq\left\|\mathcal{P}_{\bar{T}^{\perp}}\left(\Delta_{L}\right)\right\|_{b(\rho)}+\left\|\mathcal{P}_{\bar{T}}\left(\Delta_{L}\right)\right\|_{b(\rho)} \\
& \leq\left\|\mathcal{P}_{\bar{T}^{\perp}}\left(\Delta_{L}\right)\right\|_{*}+\min \left\{\left\|\mathcal{P}_{\bar{T}}\left(\Delta_{L}\right)\right\|_{*},\left\|\mathcal{P}_{\bar{T}}\left(\Delta_{L}\right)\right\|_{b(\rho)}\right\} \quad \text { (Lemma 3) } \\
\left\|\Delta_{L}\right\|_{*} & \leq\left\|\mathcal{P}_{\bar{T}^{\perp}}\left(\Delta_{L}\right)\right\|_{*}+\left\|\mathcal{P}_{\bar{T}}\left(\Delta_{L}\right)\right\|_{*}
\end{aligned}
$$

and Lemma 15
Note that we have an error bound for $\Delta_{L}$ in $\|\cdot\|_{b(\rho)}$ norm, which can be significantly smaller than the bound for the trace norm of $\Delta_{L}$.

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[^0]:    ${ }^{1}$ Chandrasekaran et al. (2009) do not explicitly work out the non-square case, but claim that $n$ can be replaced in their analysis by the larger matrix dimension $\max \{m, n\}$. However this does not seem possible, and the analysis there should only lead to the quite suboptimal dimensionality dependency $\min \{m, n\}$. This is because a rectangular matrix $\bar{X}_{L}$ will have left and right singular vectors of different dimensions and thus different allowable ranges of infinity norms.

