# Analysis of a randomized approximation scheme for matrix multiplication 

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August 27, 2018


#### Abstract

This note gives a simple analysis of a randomized approximation scheme for matrix multiplication proposed by [Sar06] based on a random rotation followed by uniform column sampling. The result follows from a matrix version of Bernstein's inequality and a tail inequality for quadratic forms in subgaussian random vectors.


## 1 Introduction

Let $A:=\left[a_{1}\left|a_{2}\right| \cdots \mid a_{m}\right] \in \mathbb{R}^{d_{A} \times m}$ and $B:=\left[b_{1}\left|b_{2}\right| \cdots \mid b_{m}\right] \in \mathbb{R}^{d_{B} \times m}$ be fixed matrices, each with $m$ columns. If $m$ is very large, then the straightforward computation of the matrix product $A B^{\top}$ (with $\Omega\left(d_{A} d_{B} m\right)$ operations) can be prohibitive.

We can instead approximate the product using the following randomized scheme. Let $\Theta \in \mathbb{R}^{m \times m}$ be a random orthogonal matrix; the distribution of $\Theta$ will be specified later in Theorem 1, but a key property of $\Theta$ will be that the matrix products

$$
\tilde{A}:=A \Theta \quad \text { and } \quad \tilde{B}:=B \Theta
$$

can be computed with $O\left(\left(d_{A}+d_{B}\right) m \log m\right)$ operations. Given the products $\tilde{A}=\left[\tilde{a}_{1}\left|\tilde{a}_{2}\right| \cdots \mid \tilde{a}_{m}\right]$ and $\tilde{B}=\left[\tilde{b}_{1}\left|\tilde{b}_{2}\right| \cdots \mid \tilde{b}_{m}\right]$, we take a small uniform random sample of pairs of their columns (drawn with replacement)

$$
\left(\tilde{a}_{i_{1}}, \tilde{b}_{i_{1}}\right),\left(\tilde{a}_{i_{2}}, \tilde{b}_{i_{2}}\right), \ldots,\left(\tilde{a}_{i_{n}}, \tilde{b}_{i_{n}}\right)
$$

and then compute the sum of outer products

$$
\widehat{A B^{\top}}:=\frac{m}{n} \sum_{j=1}^{n} \tilde{a}_{i_{j}} \tilde{b}_{i_{j}}^{\top} .
$$

It is easy to check that $\widehat{A B^{\top}}$ is an unbiased estimator of $A B^{\top}$. The sum can be computed from $\tilde{A}$ and $\tilde{B}$ with $O\left(d_{A} d_{B} n\right)$ operations, so overall, the matrix $\widehat{A B^{\top}}$ can be computed with $O\left(d_{A} d_{B} n+\right.$

[^0]$\left(d_{A}+d_{B}\right) m \log m$ ) operations. (In fact, the $\log m$ can be replaced by $\log n$ [AL08].) Therefore, we would like $n$ to be as small as possible so that, with high probability, $\left\|\widehat{A B^{\top}}-A B^{\top}\right\| \leq \varepsilon\|A\|\|B\|$ for some error $\varepsilon>0$, where $\|\cdot\|$ denotes the spectral norm. As shown in Theorem 1, it suffices to have
$$
n=\Omega\left(\frac{(k+\log (m)) \log (k)}{\varepsilon^{2}}\right)
$$
where $k:=\max \left\{\operatorname{tr}\left(A^{\top} A\right) /\|A\|^{2}, \operatorname{tr}\left(B^{\top} B\right) /\|B\|^{2}\right\} \leq \max \{\operatorname{rank}(A), \operatorname{rank}(B)\}$.
A flawed analysis of a different scheme based on non-uniform column sampling (without a random rotation $\Theta$ ) was given in [HKZ12a]; that analysis gave an incorrect bound on $\left\|\mathbb{E}\left[X^{2}\right]\right\|$ for a certain random symmetric matrix $X$. A different analysis of this non-uniform sampling scheme can be found in [MZ11], but that analysis has some deficiencies as pointed out in [HKZ12a]. The scheme studied in the present work, which employs a certain random rotation followed by uniform column sampling, was proposed by [Sar06], and is based on the Fast Johnson-Lindenstrauss Transform of [AC09]. The analysis given in [Sar06] bounds the Frobenius norm error; in this work, we bound the spectral norm error. A similar but slightly looser analysis of spectral norm error was very recently provided in [ABTZ12].

## 2 Analysis

Let $[m]:=\{1,2, \ldots, m\}$.
Theorem 1. Pick any $\delta \in(0,1 / 3)$, and let $k:=\max \left\{\operatorname{tr}\left(A A^{\top}\right) /\|A\|^{2}, \operatorname{tr}\left(B B^{\top}\right) /\|B\|^{2}\right\}$ (note that $k \leq \max \{\operatorname{rank}(A), \operatorname{rank}(B)\}$. Assume $\Theta=\frac{1}{\sqrt{m}} D H$, where $D=\operatorname{diag}(\epsilon), \epsilon \in\{ \pm 1\}^{m}$ is a vector of independent Rademacher random variables, and $H \in\{ \pm 1\}^{m \times m}$ is a Hadamard matrix. With probability at least $1-\delta$,

$$
\left.\begin{array}{rl}
\left\|\widehat{A B^{\top}}-A B^{\top}\right\| \leq\|A\|\|B\|\left(\sqrt{\frac{4(k+2 \sqrt{k \ln (3 m / \delta)}+2 \ln (3 m / \delta)+1) \ln (6 k / \delta)}{n}}\right. \\
& +\frac{2(k+2 \sqrt{k \ln (3 m / \delta)}+2 \ln (3 m / \delta)+1) \ln (6 k / \delta)}{3 n}
\end{array}\right) .
$$

The proof of Theorem 1 is a consequence of the following lemmas, combined with a union bound. Lemma 1 bounds the error in terms of a certain quantity $\mu$ which depends on the random orthogonal matrix $\Theta$ (and $A$ and $B$ ). Lemma 2 gives a bound on $\mu$ that holds with high probability over the random choice of $\Theta$.

Lemma 1. Define $Q=\left[q_{1}\left|q_{2}\right| \cdots \mid q_{m}\right]:=\|A\|^{-1} A \Theta, R=\left[r_{1}\left|r_{2}\right| \cdots \mid r_{m}\right]:=\|B\|^{-1} B \Theta, k_{A}:=$ $\operatorname{tr}\left(Q Q^{\top}\right)=\operatorname{tr}\left(A A^{\top}\right) /\|A\|^{2}, k_{B}:=\operatorname{tr}\left(R R^{\top}\right)=\operatorname{tr}\left(B B^{\top}\right) /\|B\|^{2}$, and

$$
\mu:=m \max \left(\left\{\left\|q_{i}\right\|^{2}: i \in[m]\right\} \cup\left\{\left\|r_{i}\right\|^{2}: i \in[m]\right\}\right) .
$$

Then

$$
\operatorname{Pr}\left[\left\|\widehat{A B^{\top}}-A B^{\top}\right\|>\|A\|\|B\|\left(\sqrt{\frac{2(\mu+1) t}{n}}+\frac{(\mu+1) t}{3 n}\right)\right] \leq 2 \sqrt{k_{A} k_{B}} \cdot \frac{t}{e^{t}-t-1} .
$$

Proof. Observe that because $\Theta$ is orthogonal,

$$
\left\|\widehat{A B^{\top}}-A B^{\top}\right\|=\|A\|\|B\|\left\|\frac{m}{n} \sum_{j=1}^{n} q_{i_{j}} r_{i_{j}}^{\top}-Q R^{\top}\right\|
$$

We now derive a high probability bound for the last term on the right-hand side. Define a random symmetric matrix $X$ with

$$
\operatorname{Pr}\left[X=m\left[\begin{array}{cc}
0 & q_{i} r_{i}^{\top} \\
r_{i} q_{i}^{\top} & 0
\end{array}\right]\right]=\frac{1}{m}, \quad i \in[m],
$$

and let $X_{1}, X_{2}, \ldots, X_{n}$ be independent copies of $X$. Define

$$
\widehat{M}:=\frac{1}{n} \sum_{j=1}^{n} X_{j} \quad \text { and } \quad M:=\left[\begin{array}{cc}
0 & Q R^{\top} \\
R Q^{\top} & 0
\end{array}\right] .
$$

Then

$$
\|\widehat{M}-M\|=\left\|\frac{1}{n} \sum_{j=1}^{n}\left(X_{j}-M\right)\right\| \text { distribution }\left\|\frac{m}{n} \sum_{j=1}^{n} q_{i_{j}} r_{i_{j}}^{\top}-Q R^{\top}\right\| .
$$

Observe that $\mathbb{E}[X-M]=0$ and $\|X-M\| \leq\|X\|+\|M\| \leq \mu+1$. Moreover,

$$
\begin{aligned}
\mathbb{E}[X]^{2} & =M^{2}=\left[\begin{array}{cc}
Q R^{\top} R Q^{\top} & 0 \\
0 & R Q^{\top} Q R^{\top}
\end{array}\right], \\
\mathbb{E}\left[X^{2}\right] & =\sum_{i=1}^{m} m\left[\begin{array}{cc}
\left\|r_{i}\right\|^{2} q_{i} q_{i}^{\top} & 0 \\
0 & \left\|q_{i}\right\|^{2} r_{i} r_{i}^{\top}
\end{array}\right]=m\left[\begin{array}{cc}
\sum_{i=1}^{m}\left\|r_{i}\right\|^{2} q_{i} q_{i}^{\top} & 0 \\
0 & \sum_{i=1}^{m}\left\|q_{i}\right\|^{2} r_{i} r_{i}^{\top}
\end{array}\right], \\
\operatorname{tr}\left(\mathbb{E}\left[X^{2}\right]\right) & =2 m \sum_{i=1}^{m}\left\|q_{i}\right\|^{2}\left\|r_{i}\right\|^{2} \leq 2 \mu \sum_{i=1}^{m}\left\|q_{i}\right\|\left\|r_{i}\right\| \leq 2 \mu \sqrt{k_{A} k_{B}}, \\
\left\|\mathbb{E}\left[X^{2}\right]\right\| & \leq m \max \left\{\left\|\sum_{i=1}^{m}\right\| r_{i}\left\|^{2} q_{i} q_{i}^{\top}\right\|,\left\|\sum_{i=1}^{m}\right\| q_{i}\left\|^{2} r_{i} r_{i}^{\top}\right\|\right\} \leq \mu \max \left\{\left\|Q Q^{\top}\right\|,\left\|R R^{\top}\right\|\right\}=\mu, \\
\left\|\mathbb{E}\left[(X-M)^{2}\right]\right\| & =\left\|\mathbb{E}\left[X^{2}\right]-M^{2}\right\| \leq \mu+1 .
\end{aligned}
$$

Therefore, by the matrix Bernstein inequality from [HKZ12a],

$$
\operatorname{Pr}\left[\|\widehat{M}-M\|>\sqrt{\frac{2(\mu+1) t}{n}}+\frac{(\mu+1) t}{3 n}\right] \leq 2 \sqrt{k_{A} k_{B}} \cdot \frac{t}{e^{t}-t-1} .
$$

The lemma follows.
The following lemma is a special case of a result found in [HKZ11].
Lemma 2. Assume $\Theta=\frac{1}{\sqrt{m}} D H$, where $D=\operatorname{diag}(\epsilon), \epsilon \in\{ \pm 1\}^{m}$ is a vector of independent Rademacher random variables, and $H \in\{ \pm 1\}^{m \times m}$ is a Hadamard matrix. Let $Z \in \mathbb{R}^{m \times d}$ be a matrix with $\|Z\| \leq 1$, and set $k_{Z}:=\operatorname{tr}\left(Z Z^{\top}\right)$. Then

$$
\operatorname{Pr}\left[\max \left\{\left\|Z^{\top} \Theta e_{i}\right\|^{2}: i \in[m]\right\}>\frac{k_{Z}+2 \sqrt{k_{Z}(\ln (m)+t)}+2(\ln (m)+t)}{m}\right] \leq e^{-t}
$$

where $e_{i} \in\{0,1\}^{m}$ is the $i$-th coordinate axis vector in $\mathbb{R}^{m}$.

Proof. Observe that for each $i \in[m]$, the random vector $\sqrt{m} \Theta e_{i}$ has the same distribution as $\epsilon$. Moreover, $\epsilon$ is a subgaussian random vector in the sense that $\mathbb{E}\left[\exp \left(\alpha^{\top} \epsilon\right)\right] \leq \exp \left(\|\alpha\|^{2} / 2\right)$ for any vector $\alpha \in \mathbb{R}^{m}$. Therefore, we may apply a tail bound for quadratic forms in subgaussian random vectors (e.g., [HKZ12b]) to obtain

$$
\operatorname{Pr}\left[\left\|\sqrt{m} Z^{\top} \Theta e_{i}\right\|^{2}>\operatorname{tr}\left(Z Z^{\top}\right)+2 \sqrt{\operatorname{tr}\left(\left(Z Z^{\top}\right)^{2}\right) \tau}+2\left\|Z Z^{\top}\right\| \tau\right] \leq e^{-\tau}
$$

for each $i \in[m]$ and any $\tau>0$. The lemma follows by observing that $\left\|Z Z^{\top}\right\| \leq 1$ and $\operatorname{tr}\left(\left(Z Z^{\top}\right)^{2}\right) \leq$ $\operatorname{tr}\left(Z Z^{\top}\right)\left\|Z Z^{\top}\right\| \leq k_{Z}$, and applying a union bound over all $i \in[m]$.

We note that Lemma 2 holds for many other distributions of orthogonal matrices (with possibly worse constants). All that is required is that $\sqrt{m} \Theta e_{i}$ be a subgaussian random vector for each $i \in[m]$. See [HKZ11] for more discussion.

Proof of Theorem 1. We apply Lemma 2 with both $Z=A /\|A\|$ and $Z=B /\|B\|$, and combine the implied probability bounds with a union bound to obtain

$$
\operatorname{Pr}[\mu>k+2 \sqrt{k \log (3 m / \delta)}+2 \ln (3 m / \delta)] \leq 2 \delta / 3,
$$

where $\mu$ is defined in the statement of Lemma 1 , and the probabiltiy is taken with respect to the random choice of $\Theta$. Now we apply Lemma 1 , together with the bound $t /\left(e^{t}-t-1\right) \leq e^{-t / 2}$ for $t \geq 2.6$, and substitute $t:=2 \ln (6 k / \delta)$ to obtain

$$
\operatorname{Pr}\left[\left\|\widehat{A B^{\top}}-A B^{\top}\right\|>\|A\|\|B\|\left(\sqrt{\frac{4(\mu+1) \ln (6 k / \delta)}{n}}+\frac{2(\mu+1) \ln (6 k / \delta)}{3 n}\right)\right] \leq \delta / 3 .
$$

Combining the two probability bounds with a union bound implies the claim.

## Acknowledgements

We thank Joel Tropp for pointing out the error in the analysis from [HKZ12a].

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