Analysis of a randomized approximation scheme for matrix multiplication

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Abstract

This note gives a simple analysis of a randomized approximation scheme for matrix multiplication proposed by [Sar06] based on a random rotation followed by uniform column sampling. The result follows from a matrix version of Bernstein's inequality and a tail inequality for quadratic forms in subgaussian random vectors.

1 Introduction

Let $A := [a_1|a_2|\cdots|a_m] \in \mathbb{R}^{d_A \times m}$ and $B := [b_1|b_2|\cdots|b_m] \in \mathbb{R}^{d_B \times m}$ be fixed matrices, each with m columns. If m is very large, then the straightforward computation of the matrix product AB^{\top} (with $\Omega(d_A d_B m)$ operations) can be prohibitive.

We can instead approximate the product using the following randomized scheme. Let $\Theta \in \mathbb{R}^{m \times m}$ be a random orthogonal matrix; the distribution of Θ will be specified later in Theorem 1, but a key property of Θ will be that the matrix products

 $\tilde{A} := A\Theta$ and $\tilde{B} := B\Theta$

can be computed with $O((d_A + d_B)m \log m)$ operations. Given the products $\tilde{A} = [\tilde{a}_1|\tilde{a}_2|\cdots|\tilde{a}_m]$ and $\tilde{B} = [\tilde{b}_1|\tilde{b}_2|\cdots|\tilde{b}_m]$, we take a small uniform random sample of pairs of their columns (drawn with replacement)

$$(\tilde{a}_{i_1}, \tilde{b}_{i_1}), (\tilde{a}_{i_2}, \tilde{b}_{i_2}), \dots, (\tilde{a}_{i_n}, \tilde{b}_{i_n}),$$

and then compute the sum of outer products

$$\widehat{AB^{\top}} := \frac{m}{n} \sum_{j=1}^{n} \tilde{a}_{i_j} \tilde{b}_{i_j}^{\top}.$$

It is easy to check that $\widehat{AB^{\top}}$ is an unbiased estimator of AB^{\top} . The sum can be computed from \tilde{A} and \tilde{B} with $O(d_A d_B n)$ operations, so overall, the matrix $\widehat{AB^{\top}}$ can be computed with $O(d_A d_B n + C)$

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 $(d_A + d_B)m \log m)$ operations. (In fact, the $\log m$ can be replaced by $\log n$ [AL08].) Therefore, we would like n to be as small as possible so that, with high probability, $\|\widehat{AB^{\top}} - AB^{\top}\| \leq \varepsilon \|A\| \|B\|$ for some error $\varepsilon > 0$, where $\|\cdot\|$ denotes the spectral norm. As shown in Theorem 1, it suffices to have

$$n = \Omega\left(\frac{(k + \log(m))\log(k)}{\varepsilon^2}\right),$$

where $k := \max\{\operatorname{tr}(A^{\top}A)/\|A\|^2, \ \operatorname{tr}(B^{\top}B)/\|B\|^2\} \le \max\{\operatorname{rank}(A), \operatorname{rank}(B)\}.$

A flawed analysis of a different scheme based on non-uniform column sampling (without a random rotation Θ) was given in [HKZ12a]; that analysis gave an incorrect bound on $||\mathbb{E}[X^2]||$ for a certain random symmetric matrix X. A different analysis of this non-uniform sampling scheme can be found in [MZ11], but that analysis has some deficiencies as pointed out in [HKZ12a]. The scheme studied in the present work, which employs a certain random rotation followed by uniform column sampling, was proposed by [Sar06], and is based on the Fast Johnson-Lindenstrauss Transform of [AC09]. The analysis given in [Sar06] bounds the Frobenius norm error; in this work, we bound the spectral norm error. A similar but slightly looser analysis of spectral norm error was very recently provided in [ABTZ12].

2 Analysis

Let $[m] := \{1, 2, \dots, m\}.$

Theorem 1. Pick any $\delta \in (0, 1/3)$, and let $k := \max\{\operatorname{tr}(AA^{\top})/||A||^2, \operatorname{tr}(BB^{\top})/||B||^2\}$ (note that $k \leq \max\{\operatorname{rank}(A), \operatorname{rank}(B)\}$. Assume $\Theta = \frac{1}{\sqrt{m}}DH$, where $D = \operatorname{diag}(\epsilon), \epsilon \in \{\pm 1\}^m$ is a vector of independent Rademacher random variables, and $H \in \{\pm 1\}^{m \times m}$ is a Hadamard matrix. With probability at least $1 - \delta$,

$$\begin{split} \|\widehat{AB^{\top}} - AB^{\top}\| &\leq \|A\| \|B\| \left(\sqrt{\frac{4(k + 2\sqrt{k\ln(3m/\delta)} + 2\ln(3m/\delta) + 1)\ln(6k/\delta)}{n}} \\ &+ \frac{2(k + 2\sqrt{k\ln(3m/\delta)} + 2\ln(3m/\delta) + 1)\ln(6k/\delta)}{3n} \right). \end{split}$$

The proof of Theorem 1 is a consequence of the following lemmas, combined with a union bound. Lemma 1 bounds the error in terms of a certain quantity μ which depends on the random orthogonal matrix Θ (and A and B). Lemma 2 gives a bound on μ that holds with high probability over the random choice of Θ .

Lemma 1. Define $Q = [q_1|q_2|\cdots|q_m] := ||A||^{-1}A\Theta, R = [r_1|r_2|\cdots|r_m] := ||B||^{-1}B\Theta, k_A := \operatorname{tr}(QQ^{\top}) = \operatorname{tr}(AA^{\top})/||A||^2, k_B := \operatorname{tr}(RR^{\top}) = \operatorname{tr}(BB^{\top})/||B||^2, and$ $\mu := m \max\left(\{||q_i||^2 : i \in [m]\} \cup \{||r_i||^2 : i \in [m]\}\}\right).$

Then

$$\Pr\left[\|\widehat{AB^{\top}} - AB^{\top}\| > \|A\| \|B\| \left(\sqrt{\frac{2(\mu+1)t}{n}} + \frac{(\mu+1)t}{3n}\right)\right] \le 2\sqrt{k_A k_B} \cdot \frac{t}{e^t - t - 1}.$$

Proof. Observe that because Θ is orthogonal,

$$\|\widehat{AB^{\top}} - AB^{\top}\| = \|A\| \|B\| \left\| \frac{m}{n} \sum_{j=1}^{n} q_{i_j} r_{i_j}^{\top} - QR^{\top} \right\|.$$

We now derive a high probability bound for the last term on the right-hand side. Define a random symmetric matrix X with

$$\Pr\left[X = m \begin{bmatrix} 0 & q_i r_i^\top \\ r_i q_i^\top & 0 \end{bmatrix}\right] = \frac{1}{m}, \quad i \in [m],$$

and let X_1, X_2, \ldots, X_n be independent copies of X. Define

$$\widehat{M} := \frac{1}{n} \sum_{j=1}^{n} X_j$$
 and $M := \begin{bmatrix} 0 & QR^{\top} \\ RQ^{\top} & 0 \end{bmatrix}$.

Then

$$\|\widehat{M} - M\| = \left\|\frac{1}{n}\sum_{j=1}^{n} (X_j - M)\right\| \stackrel{\text{distribution}}{=} \left\|\frac{m}{n}\sum_{j=1}^{n} q_{ij}r_{ij}^{\top} - QR^{\top}\right\|.$$

Observe that $\mathbb{E}[X - M] = 0$ and $||X - M|| \le ||X|| + ||M|| \le \mu + 1$. Moreover,

$$\begin{split} \mathbb{E}[X]^2 &= M^2 = \begin{bmatrix} QR^{^{\top}}RQ^{^{\top}} & 0\\ 0 & RQ^{^{\top}}QR^{^{\top}} \end{bmatrix},\\ \mathbb{E}[X^2] &= \sum_{i=1}^m m \begin{bmatrix} \|r_i\|^2 q_i q_i^{^{\top}} & 0\\ 0 & \|q_i\|^2 r_i r_i^{^{\top}} \end{bmatrix} = m \begin{bmatrix} \sum_{i=1}^m \|r_i\|^2 q_i q_i^{^{\top}} & 0\\ 0 & \sum_{i=1}^m \|q_i\|^2 r_i r_i^{^{\top}} \end{bmatrix},\\ \mathrm{tr}(\mathbb{E}[X^2]) &= 2m \sum_{i=1}^m \|q_i\|^2 \|r_i\|^2 \le 2\mu \sum_{i=1}^m \|q_i\| \|r_i\| \le 2\mu \sqrt{k_A k_B},\\ \|\mathbb{E}[X^2]\| \le m \max\Big\{\Big\|\sum_{i=1}^m \|r_i\|^2 q_i q_i^{^{\top}}\Big\|, \ \Big\|\sum_{i=1}^m \|q_i\|^2 r_i r_i^{^{\top}}\Big\|\Big\} \le \mu \max\Big\{\|QQ^{^{\top}}\|, \ \|RR^{^{\top}}\|\Big\} = \mu (X-M)^2]\| = \|\mathbb{E}[X^2] - M^2\| \le \mu + 1. \end{split}$$

Therefore, by the matrix Bernstein inequality from [HKZ12a],

$$\Pr\left[\|\widehat{M} - M\| > \sqrt{\frac{2(\mu+1)t}{n}} + \frac{(\mu+1)t}{3n}\right] \le 2\sqrt{k_A k_B} \cdot \frac{t}{e^t - t - 1}.$$

The lemma follows.

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The following lemma is a special case of a result found in [HKZ11].

Lemma 2. Assume $\Theta = \frac{1}{\sqrt{m}}DH$, where $D = \text{diag}(\epsilon)$, $\epsilon \in \{\pm 1\}^m$ is a vector of independent Rademacher random variables, and $H \in \{\pm 1\}^{m \times m}$ is a Hadamard matrix. Let $Z \in \mathbb{R}^{m \times d}$ be a matrix with $\|Z\| \leq 1$, and set $k_Z := \text{tr}(ZZ^{\top})$. Then

$$\Pr\left[\max\{\|Z^{\top}\Theta e_i\|^2: i \in [m]\} > \frac{k_Z + 2\sqrt{k_Z(\ln(m) + t)} + 2(\ln(m) + t)}{m}\right] \le e^{-t}$$

where $e_i \in \{0,1\}^m$ is the *i*-th coordinate axis vector in \mathbb{R}^m .

Proof. Observe that for each $i \in [m]$, the random vector $\sqrt{m}\Theta e_i$ has the same distribution as ϵ . Moreover, ϵ is a subgaussian random vector in the sense that $\mathbb{E}[\exp(\alpha^{\top}\epsilon)] \leq \exp(\|\alpha\|^2/2)$ for any vector $\alpha \in \mathbb{R}^m$. Therefore, we may apply a tail bound for quadratic forms in subgaussian random vectors (*e.g.*, [HKZ12b]) to obtain

$$\Pr\left[\|\sqrt{m}Z^{\top}\Theta e_i\|^2 > \operatorname{tr}(ZZ^{\top}) + 2\sqrt{\operatorname{tr}((ZZ^{\top})^2)\tau} + 2\|ZZ^{\top}\|\tau\right] \le e^{-\tau}$$

for each $i \in [m]$ and any $\tau > 0$. The lemma follows by observing that $||ZZ^{\top}|| \leq 1$ and $\operatorname{tr}((ZZ^{\top})^2) \leq \operatorname{tr}(ZZ^{\top})||ZZ^{\top}|| \leq k_Z$, and applying a union bound over all $i \in [m]$.

We note that Lemma 2 holds for many other distributions of orthogonal matrices (with possibly worse constants). All that is required is that $\sqrt{m}\Theta e_i$ be a subgaussian random vector for each $i \in [m]$. See [HKZ11] for more discussion.

Proof of Theorem 1. We apply Lemma 2 with both Z = A/||A|| and Z = B/||B||, and combine the implied probability bounds with a union bound to obtain

$$\Pr\left[\mu > k + 2\sqrt{k \log(3m/\delta)} + 2\ln(3m/\delta)\right] \le 2\delta/3,$$

where μ is defined in the statement of Lemma 1, and the probability is taken with respect to the random choice of Θ . Now we apply Lemma 1, together with the bound $t/(e^t - t - 1) \leq e^{-t/2}$ for $t \geq 2.6$, and substitute $t := 2 \ln(6k/\delta)$ to obtain

$$\Pr\left[\|\widehat{AB^{\top}} - AB^{\top}\| > \|A\| \|B\| \left(\sqrt{\frac{4(\mu+1)\ln(6k/\delta)}{n}} + \frac{2(\mu+1)\ln(6k/\delta)}{3n}\right)\right] \le \delta/3.$$

Combining the two probability bounds with a union bound implies the claim.

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