A Markov Chain Theory Approach to Characterizing the Minimax Optimality of Stochastic Gradient Descent (for Least Squares)

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Abstract

This work provides a simplified proof of the statistical minimax optimality of (iterate averaged) stochastic gradient descent (SGD), for the special case of least squares. This result is obtained by analyzing SGD as a stochastic process and by sharply characterizing the stationary covariance matrix of this process. The finite rate optimality characterization captures the constant factors and addresses model mis-specification.

1 Introduction

Stochastic gradient descent is among the most commonly used practical algorithms for large scale stochastic optimization. The seminal result of [9, 8] formalized this effectiveness, showing that for certain (locally quadric) problems, asymptotically, stochastic gradient descent is statistically minimax optimal (provided the iterates are averaged). There are a number of more modern proofs [1, 3, 2, 5] of this fact, which provide finite rates of convergence. Other recent algorithms also achieve the statistically optimal minimax rate, with finite convergence rates [4].

This work provides a short proof of this minimax optimality for SGD for the special case of least squares through a characterization of SGD as a stochastic process. The proof builds on ideas developed in [2, 5].

SGD for least squares. The expected square loss for $w \in \mathbb{R}^d$ over input-output pairs (x, y), where $x \in \mathbb{R}^d$ and $y \in \mathbb{R}$ are sampled from a distribution \mathcal{D} , is:

$$L(w) = \frac{1}{2} \mathbb{E}_{(x,y) \sim \mathcal{D}}[(y - w \cdot x)^{2}]$$

The optimal weight is denoted by:

$$w^* := \underset{w}{\operatorname{argmin}} L(w)$$
.

Assume the argmin in unique.

Stochastic gradient descent proceeds as follows: at each iteration t, using an i.i.d. sample $(x_t, y_t) \sim \mathcal{D}$, the update of w_t is:

$$w_t = w_{t-1} + \gamma (y_t - w_{t-1} \cdot x_t) x_t$$

where γ is a fixed stepsize.

Notation. For a symmetric positive definite matrix A and a vector x, define:

$$||x||_A^2 := x^\top A x.$$

For a symmetric matrix M, define the induced matrix norm under A as:

$$||M||_A := \max_{||v||=1} \frac{v^\top M v}{v^\top A v} = ||A^{-1/2} M A^{-1/2}||.$$

The statistically optimal rate. Using n samples (and for large enough n), the minimax optimal rate is achieved by the maximum likelihood estimator (the MLE), or, equivalently, the empirical risk minimizer. Given n i.i.d. samples $\{(x_i, y_i)\}_{i=1}^n$, define

$$\widehat{w}_{n}^{\text{MLE}} := \arg\min_{w} \frac{1}{n} \sum_{i=1}^{n} \frac{1}{2} (y_{i} - w \cdot x_{i})^{2}$$

where $\widehat{w}_n^{\text{MLE}}$ denotes the MLE estimator over the *n* samples.

This rate can be characterized as follows: define

$$\sigma_{\text{MLE}}^2 := \frac{1}{2} \mathbb{E} \left[(y - w^* x)^2 ||x||_{H^{-1}}^2 \right],$$

and the (asymptotic) rate of the MLE is σ_{MLE}^2/n [7, 10]. Precisely,

$$\lim_{n \to \infty} \frac{\mathbb{E}[L(\widehat{w}_n^{\mathrm{MLE}})] - L(w^*)}{\sigma_{\mathrm{MLE}}^2/n} = 1,$$

The works of [9, 8] proved that a certain averaged stochastic gradient method achieves this minimax rate, in the limit.

For the case of additive noise models (i.e. the "well-specified" case), the assumption is that $y = w^* \cdot x + \eta$, with η being independent of x. Here, it is straightforward to see that:

$$\frac{\sigma_{\text{MLE}}^2}{n} = \frac{1}{2} \frac{d\sigma^2}{n}.$$

The rate of σ_{MLE}^2/n is still minimax optimal even among mis-specified models, where the additive noise assumption may not hold [6, 7, 10].

Assumptions. Assume the fourth moment of x is finite. Denote the second moment matrix of x as

$$H := \mathbb{E}[xx^\top]\,,$$

and suppose H is strictly positive definite with minimal eigenvalue:

$$\mu := \sigma_{\min}(H)$$
.

Define \mathbb{R}^2 as the smallest value which satisfies:

$$\mathbb{E}[\|x\|^2 x x^\top] \leq R^2 \mathbb{E}[x x^\top].$$

This implies $Tr(H) = \mathbb{E}||x||^2 \le R^2$.

2 Statistical Risk Bounds

Define:

$$\Sigma := \mathbb{E}[(y - w^*x)^2 x x^\top],$$

and so the optimal constant in the rate can be written as:

$$\sigma_{\text{MLE}}^2 = \frac{1}{2} \text{Tr}(H^{-1}\Sigma) = \frac{1}{2} \mathbb{E} \left[(y - w^* x)^2 ||x||_{H^{-1}}^2 \right],$$

For the mis-specified case, it is helpful to define:

$$\rho_{\text{misspec}} := \frac{d\|\Sigma\|_H}{\text{Tr}(H^{-1}\Sigma)},$$

which can be viewed as a measure of how mis-specified the model is. Note if the model is well-specified, then $\rho_{\text{misspec}} = 1$.

Denote the average iterate, averaged from iteration t to T, by:

$$\overline{w}_{t:T} := \frac{1}{T-t} \sum_{t'=t}^{T-1} w_{t'}.$$

Theorem 1. Suppose $\gamma < \frac{1}{R^2}$. The risk is bounded as:

$$\mathbb{E}[L(\overline{w}_{t:T})] - L(w^*) \le \left(\sqrt{\frac{1}{2}} \exp\left(-\gamma \mu t\right) R^2 \|w_0 - w^*\|^2 + \sqrt{\left(1 + \frac{\gamma R^2}{1 - \gamma R^2} \rho_{\text{misspec}}\right) \frac{\sigma_{\text{MLE}}^2}{T - t}}\right)^2.$$

The bias term (the first term) decays at a geometric rate (one can set t=T/2 or maintain multiple running averages if T is not known in advance). If $\gamma = 1/(2R^2)$ and the model is well-specified ($\rho_{\rm misspec} = 1$), then the variance term is $2\sigma_{\rm MLE}/\sqrt{T-t}$, and the rate of the bias contraction is μ/R^2 . If the model is not well specified, then using a smaller stepsize of $\gamma = 1/(2\rho_{\rm misspec}R^2)$, leads to the same minimax optimal rate (up to a constant factor of 2), albeit at a slower bias contraction rate. In the mis-specified case, an example in [5] shows that such a smaller stepsize is required in order to be within a constant factor of the minimax rate. An even smaller stepsize leads to a constant even closer to that of the optimal rate.

3 Analysis

The analysis first characterizes a bias/variance decomposition, where the variance is bounded in terms of properties of the stationary covariance of w_t . Then this asymptotic covariance matrix is analyzed.

Throughout assume:

$$\gamma < \frac{1}{R^2} \, .$$

3.1 The Bias-Variance Decomposition

The gradient at w^* in iteration t is:

$$\xi_t := -(y_t - w^* \cdot x_t) x_t \,,$$

which is a mean 0 quantity. Also define:

$$B_t := \mathbf{I} - x_t x_t^{\top} .$$

The update rule can be written as:

$$w_{t} - w^{*} = w_{t-1} - w^{*} + \gamma (y_{t} - w_{t-1} \cdot x_{t}) x_{t}$$
$$= (I - \gamma x_{t} x_{t}^{\top}) (w_{t-1} - w^{*}) - \gamma \xi_{t}$$
$$= B_{t} (w_{t-1} - w^{*}) - \gamma \xi_{t}.$$

Roughly speaking, the above shows how the process on $w_t - w^*$ consists of a contraction along with an addition of a zero mean quantity.

From recursion,

$$w_t - w^* = B_t \cdots B_1(w_0 - w^*) - \gamma \left(\xi_t + B_t \xi_{t-1} + \cdots + B_t \cdots B_2 \xi_1\right). \tag{1}$$

It is helpful to consider a certain bias and variance decomposition. Let us write:

$$\mathbb{E}[\|\overline{w}_{t:T} - w^*\|_H^2 | \xi_0 = \dots = \xi_T = 0] := \frac{1}{(T - t)^2} \mathbb{E}\left[\left\| \sum_{\tau = t}^{T - 1} B_\tau \cdots B_1(w_0 - w^*) \right\|_H^2 \right].$$

and

$$\mathbb{E}[\|\overline{w}_{t:T} - w^*\|_H^2 | w_0 = w^*] = \left(\frac{\gamma}{T - t}\right)^2 \cdot \mathbb{E}\left[\left\|\sum_{\tau = t}^{T - 1} \left(\xi_\tau + B_\tau \xi_{\tau - 1} + \dots + B_\tau \dots B_2 \xi_1\right)\right\|_H^2\right]$$

(The first conditional expectation notation slightly abuses notation, and should be taken as a definition¹).

Lemma 1. The error is bounded as:

$$\mathbb{E}[L(\overline{w}_{t:T})] - L(w^*) \le \frac{1}{2} \left(\sqrt{\mathbb{E}[\|\overline{w}_{t:T} - w^*\|_H^2 | \xi_0 = \dots = \xi_T = 0]} + \sqrt{\mathbb{E}[\|\overline{w}_{t:T} - w^*\|_H^2 | w_0 = w^*]} \right)^2.$$

Proof. Equation 1 implies that:

$$\overline{w}_{t:T} - w^* = \frac{1}{T - t} \sum_{\tau = t}^{T - 1} B_{\tau} \cdots B_1(w_0 - w^*) - \frac{\gamma}{T - t} \sum_{\tau = t}^{T - 1} (\xi_{\tau} + B_{\tau} \xi_{\tau - 1} + \cdots + B_{\tau} \cdots B_2 \xi_1) .$$

Now observe that for vector valued random variables u and v, $(\mathbb{E}u^{\top}Hv)^2 \leq \mathbb{E}[\|u\|_H^2]\mathbb{E}[\|v\|_H^2]$ implies

$$\mathbb{E}\|u+v\|_H^2 \leq \left(\sqrt{\mathbb{E}\|u\|_H^2} + \sqrt{\mathbb{E}\|v\|_H^2}\right)^2,$$

the proof of the lemma follows by noting that $\mathbb{E}[L(\overline{w}_{t:T}) - L(w^*)] = \frac{1}{2}\mathbb{E}\|\overline{w}_{t:T} - w^*\|_H^2$.

Bias. The bias term is characterized as follows:

Lemma 2. For all t,

$$\mathbb{E}[\|w_t - w^*\|^2 | \xi_0 = \dots = \xi_T = 0] \le \exp(-\gamma \mu t) \|w_0 - w^*\|^2.$$

Proof. Assume $\xi_t = 0$ for all t. Observe:

$$\mathbb{E}\|w_{t} - w^{*}\|^{2} = \mathbb{E}\|w_{t-1} - w^{*}\|^{2} - 2\gamma(w_{t-1} - w^{*})^{\top}\mathbb{E}[xx^{\top}](w_{t-1} - w^{*}) + \gamma^{2}(w_{t-1} - w^{*})^{\top}\mathbb{E}[\|x\|^{2}xx^{\top}](w_{t-1} - w^{*})$$

$$\leq \mathbb{E}\|w_{t-1} - w^{*}\|^{2} - 2\gamma(w_{t-1} - w^{*})^{\top}H(w_{t-1} - w^{*}) + \gamma^{2}R^{2}(w_{t-1} - w^{*})^{\top}H(w_{t-1} - w^{*})$$

$$\leq \mathbb{E}\|w_{t-1} - w^{*}\|^{2} - \gamma\mathbb{E}\|w_{t-1} - w^{*}\|^{2}$$

$$\leq (1 - \gamma\mu)\mathbb{E}\|w_{t-1} - w^{*}\|^{2},$$

which completes the proof.

¹The abuse is due that the right hand side drops the conditioning.

Variance. Now suppose $w_0 = w^*$. Define the covariance matrix:

$$C_t := \mathbb{E}[(w_t - w^*)(w_t - w^*)^\top | w_0 = w^*]$$

Using the recursion, $w_t - w^* = B_t(w_{t-1} - w^*) + \gamma \xi_t$,

$$C_{t+1} = C_t - \gamma H C_t - \gamma C_t H + \gamma^2 \mathbb{E}[(x^\top C_t x) x x^\top] + \gamma^2 \Sigma$$
 (2)

which follows from:

$$\mathbb{E}[(w_t - w^*)\xi_{t+1}^{\top}] = 0$$
, and $\mathbb{E}[(x_{t+1}x_{t+1}^{\top})(w_t - w^*)\xi_{t+1}^{\top}] = 0$

(these hold since $w_t - w^*$ is mean 0 and both x_{t+1} and ξ_{t+1} are independent of $w_t - w^*$).

Lemma 3. Suppose $w_0 = w^*$. There exists a unique C_{∞} such that:

$$0 = C_0 \preceq C_1 \preceq \cdots \preceq C_{\infty}$$

where C_{∞} satisfies:

$$C_{\infty} = C_{\infty} - \gamma H C_{\infty} - \gamma C_{\infty} H + \gamma^{2} \mathbb{E}[(x^{\top} C_{\infty} x) x x^{\top}] + \gamma^{2} \Sigma.$$
(3)

Proof. By recursion,

$$w_t - w^* = B_t(w_{t-1} - w^*) + \gamma \xi_t$$

= $\gamma (\xi_t + B_t \xi_{t-1} + \dots + B_t \dots B_2 \xi_1)$.

Using that ξ_t is mean zero and independent of $B_{t'}$ and $\xi_{t'}$ for t < t',

$$C_t = \gamma^2 \left(\mathbb{E}[\xi_t \xi_t^\top] + \mathbb{E}[B_t \xi_{t-1} \xi_{t-1}^\top B_t] + \dots + \mathbb{E}[B_t \cdots B_2 \xi_1 \xi_1^\top B_2^\top \cdots B_t^\top] \right)$$

Now using that $\mathbb{E}[\xi_1 \xi_1^\top] = \Sigma$ and that ξ_t and $B_{t'}$ are independent (for $t \neq t'$),

$$C_t = \gamma^2 \left(\Sigma + \mathbb{E}[B_2 \Sigma B_2] + \dots + \mathbb{E}[B_t \dots B_2 \Sigma B_2^\top \dots B_t^\top] \right)$$

= $C_{t-1} + \gamma^2 \mathbb{E}[B_t \dots B_2 \Sigma B_2^\top \dots B_t^\top]$

which proves $C_{t-1} \leq C_t$.

To prove the limit exists, it suffices to first argue the trace of C_t is uniformly bounded from above, for all t. By taking the trace of update rule, Equation 2, for C_t ,

$$\operatorname{Tr}(C_{t+1}) = \operatorname{Tr}(C_t) - 2\gamma \operatorname{Tr}(HC_t) + \gamma^2 \operatorname{Tr}(\mathbb{E}[(x^{\top}C_tx)xx^{\top}]) + \gamma^2 \operatorname{Tr}(\Sigma).$$

Observe:

$$\operatorname{Tr}(\mathbb{E}[(x^{\top}C_tx)xx^{\top}]) = \operatorname{Tr}(\mathbb{E}[(x^{\top}C_tx)\|x\|^2]) = \operatorname{Tr}(C_t\mathbb{E}[\|x\|^2xx^{\top}]) \le R^2\operatorname{Tr}(C_tH) \tag{4}$$

and, using $\gamma \leq 1/R^2$,

$$\operatorname{Tr}(C_{t+1}) \leq \operatorname{Tr}(C_t) - \gamma \operatorname{Tr}(HC_t) + \gamma^2 \operatorname{Tr}(\Sigma) \leq (1 - \gamma \mu) \operatorname{Tr}(C_t) + \gamma^2 \operatorname{Tr}(\Sigma) \leq \frac{\gamma \operatorname{Tr}(\Sigma)}{\mu}$$
.

proving the uniform boundedness of the trace of C_t . Now, for any fixed v, the limit of $v^{\top}C_tv$ exists, by the monotone convergence theorem. From this, it follows that every entry of the matrix C_t converges.

Lemma 4. Define:

$$\overline{w}_T := \frac{1}{T} \sum_{t=0}^{T-1} w_t \,.$$

and so:

$$\frac{1}{2}\mathbb{E}[\|\overline{w}_T - w^*\|_H^2 | w_0 = w^*] \le \frac{\text{Tr}(C_\infty)}{\gamma T}$$

Proof. Note

$$\mathbb{E}[(\overline{w}_T - w^*)(\overline{w}_T - w^*)^\top | w_0 = w^*] = \frac{1}{T^2} \sum_{t=0}^{T-1} \sum_{t'=0}^{T-1} \mathbb{E}[(w_t - w^*)(w_{t'} - w^*)^\top | w_0 = w^*]$$

$$\leq \frac{1}{T^2} \sum_{t=0}^{T-1} \sum_{t'=t}^{T-1} \left(\mathbb{E}[(w_t - w^*)(w_{t'} - w^*)^\top | w_0 = w^*] + \mathbb{E}[(w_{t'} - w^*)(w_t - w^*)^\top | w_0 = w^*] \right),$$

double counting the diagonal terms $\mathbb{E}[(w_t - w^*)(w_t - w^*)^\top | w_0 = w^*] \succeq 0$. For $t \leq t'$, $\mathbb{E}[(w_{t'} - w^*)|w_0 = w^*] = (I - \gamma H)^{t'-t}\mathbb{E}[(w_t - w^*)|w_0 = w^*]$. To see why, consider the recursion $w_t - w^* = (I - \gamma x_t x_t^\top)(w_{t-1} - w^*) - \gamma \xi_t$ and take expectations to get $\mathbb{E}[w_t - w^*|w_0 = w^*] = (I - \gamma H)\mathbb{E}[w_{t-1} - w^*|w_0 = w^*]$ since the sample x_t is independent of the w_{t-1} . From this,

$$\mathbb{E}[(\overline{w}_T - w^*)(\overline{w}_T - w^*)^\top | w_0 = w^*] \leq \frac{1}{T^2} \sum_{t=0}^{T-1} \sum_{\tau=0}^{T-t-1} (I - \gamma H)^\tau C_t + C_t (I - \gamma H)^\tau,$$

and so,

$$\mathbb{E}[\|\overline{w}_{T} - w^{*}\|_{H}^{2} | w_{0} = w^{*}] = \text{Tr}(H\mathbb{E}[(\overline{w}_{T} - w^{*})(\overline{w}_{T} - w^{*})^{\top} | w_{0} = w^{*}])$$

$$\leq \frac{1}{T^{2}} \sum_{t=0}^{T-1} \sum_{\tau=0}^{T-t-1} \text{Tr}(H(\mathbf{I} - \gamma H)^{\tau} C_{t}) + \text{Tr}(C_{t}(\mathbf{I} - \gamma H)^{\tau} H).$$

Notice that $H(I - \gamma H)^{\tau} = (I - \gamma H)^{\tau} H$ for any non-negative integer τ . Since $H \succ 0$ and $I - \gamma H \succeq 0$, $H(I - \gamma H)^{\tau} \succeq 0$ because the product of two commuting PSD matrices is PSD. Also note that for PSD matrices $A, B, \text{Tr}AB \geq 0$. Hence,

$$\mathbb{E}[\|\overline{w}_{T} - w^{*}\|_{H}^{2}|w_{0} = w^{*}] \leq \frac{2}{T^{2}} \sum_{t=0}^{T-1} \sum_{\tau=0}^{\infty} \text{Tr}(H(I - \gamma H)^{\tau} C_{t})$$

$$= \frac{2}{T^{2}} \sum_{t=0}^{T-1} \text{Tr}(H(\sum_{\tau=0}^{\infty} (I - \gamma H)^{\tau}) C_{t})$$

$$= \frac{2}{T^{2}} \sum_{t=0}^{T-1} \text{Tr}(H(\gamma H)^{-1} C_{t})$$

$$= \frac{2}{\gamma T^{2}} \sum_{t=0}^{T-1} \text{Tr}(C_{t})$$

$$\leq \frac{2}{\gamma T} \cdot \text{Tr}(C_{\infty}),$$
(*)

from lemma 3 where (*) followed from

$$(\gamma H)^{-1} = (I - (I - \gamma H))^{-1} = \sum_{\tau=0}^{\infty} (I - \gamma H)^{\tau},$$

and the series converges because $I - \gamma H \prec I$.

3.2 Stationary Distribution Analysis

Define two linear operators on symmetric matrices, S and T — where S and T can be viewed as matrices acting on $\binom{d+1}{2}$ dimensions — as follows:

$$S \circ M := \mathbb{E}[(x^{\top}Mx)xx^{\top}], \qquad \mathcal{T} \circ M := HM + MH.$$

With this, C_{∞} is the solution to:

$$\mathcal{T} \circ C_{\infty} = \gamma \mathcal{S} \circ C_{\infty} + \gamma \Sigma \tag{5}$$

(due to Equation 3).

Lemma 5. (Crude C_{∞} bound) C_{∞} is bounded as:

$$C_{\infty} \preceq \frac{\gamma \|\Sigma\|_H}{1 - \gamma R^2} \mathbf{I}.$$

Proof. Define one more linear operator as follows:

$$\widetilde{\mathcal{T}} \circ M := \mathcal{T} \circ M - \gamma HMH = HM + MH - \gamma HMH$$
.

The inverse of this operator can be written as:

$$\widetilde{\mathcal{T}}^{-1} \circ M = \gamma \sum_{t=0}^{\infty} (\mathbf{I} - \gamma \widetilde{\mathcal{T}})^t \circ M = \gamma \sum_{t=0}^{\infty} (\mathbf{I} - \gamma H)^t M (\mathbf{I} - \gamma H)^t.$$

which exists since the sum converges due to fact that $0 \leq I - \gamma H \prec I$.

A few inequalities are helpful: If $0 \leq M \leq M'$, then

$$0 \preceq \widetilde{\mathcal{T}}^{-1} \circ M \preceq \widetilde{\mathcal{T}}^{-1} \circ M', \tag{6}$$

since

$$\widetilde{\mathcal{T}}^{-1} \circ M = \gamma \sum_{t=0}^{\infty} (\mathbf{I} - \gamma H)^t M (\mathbf{I} - \gamma H)^t \leq \gamma \sum_{t=0}^{\infty} (\mathbf{I} - \gamma H)^t M' (\mathbf{I} - \gamma H)^t = \widetilde{\mathcal{T}}^{-1} \circ M',$$

(which follows since $0 \leq I - \gamma H$). Also, if $0 \leq M \leq M'$, then

$$0 \le \mathcal{S} \circ M \le \mathcal{S} \circ M' \,, \tag{7}$$

which implies:

$$0 \preceq \widetilde{\mathcal{T}}^{-1} \circ \mathcal{S} \circ M \preceq \widetilde{\mathcal{T}}^{-1} \circ \mathcal{S} \circ M'. \tag{8}$$

The following inequality is also of use:

$$\Sigma \leq \|H^{-1/2}\Sigma H^{-1/2}\|H = \|\Sigma\|_H H$$
.

By definition of $\widetilde{\mathcal{T}}$,

$$\widetilde{\mathcal{T}} \circ C_{\infty} = \gamma \mathcal{S} \circ C_{\infty} + \gamma \Sigma - \gamma H C_{\infty} H.$$

Using this and Equation 6,

$$C_{\infty} = \gamma \widetilde{\mathcal{T}}^{-1} \circ \mathcal{S} \circ C_{\infty} + \gamma \widetilde{\mathcal{T}}^{-1} \circ \Sigma - \gamma \widetilde{\mathcal{T}}^{-1} \circ (HC_{\infty}H)$$

$$\preceq \gamma \widetilde{\mathcal{T}}^{-1} \circ \mathcal{S} \circ C_{\infty} + \gamma \widetilde{\mathcal{T}}^{-1} \circ \Sigma$$

$$\preceq \gamma \widetilde{\mathcal{T}}^{-1} \circ \mathcal{S} \circ C_{\infty} + \gamma \|\Sigma\|_{H} \widetilde{\mathcal{T}}^{-1} \circ H.$$

Proceeding recursively by using Equation 8,

$$C_{\infty} \leq (\gamma \widetilde{\mathcal{T}}^{-1} \circ \mathcal{S})^{2} \circ C_{\infty} + \gamma \|\Sigma\|_{H} (\gamma \widetilde{\mathcal{T}}^{-1} \circ \mathcal{S}) \circ \widetilde{\mathcal{T}}^{-1} \circ H + \gamma \|\Sigma\|_{H} \widetilde{\mathcal{T}}^{-1} \circ H$$

$$\leq \gamma \|\Sigma\|_{H} \sum_{t=0}^{\infty} (\gamma \widetilde{\mathcal{T}}^{-1} \circ \mathcal{S})^{t} \circ \widetilde{\mathcal{T}}^{-1} \circ H.$$

Using

$$S \circ I \prec R^2 H$$

and

$$\begin{split} &\widetilde{\mathcal{T}}^{-1} \circ H \\ &= \gamma \sum_{t=0}^{\infty} (\mathbf{I} - \gamma H)^{2t} H = \gamma \sum_{t=0}^{\infty} (\mathbf{I} - \gamma 2 H + \gamma^2 H)^t H \preceq \gamma \sum_{t=0}^{\infty} (\mathbf{I} - \gamma H)^t H = \gamma (\gamma H)^{-1} H = \mathbf{I} \end{split}$$

leads to

$$C_{\infty} \preceq \gamma \|\Sigma\|_H \sum_{t=0}^{\infty} (\gamma R^2)^t \mathbf{I} = \frac{\gamma \|\Sigma\|_H}{1 - \gamma R^2} \mathbf{I},$$

which completes the proof.

Lemma 6. (Refined C_{∞} bound) The $Tr(C_{\infty})$ is bounded as:

$$\operatorname{Tr}(C_{\infty}) \le \frac{\gamma}{2} \operatorname{Tr}(H^{-1}\Sigma) + \frac{1}{2} \frac{\gamma^2 R^2}{1 - \gamma R^2} d\|\Sigma\|_{H}$$

Proof. From Lemma 5 and Equation 7,

$$S \circ C_{\infty} \preceq \frac{\gamma \|\Sigma\|_H}{1 - \gamma R^2} S \circ I \preceq \frac{\gamma R^2 \|\Sigma\|_H}{1 - \gamma R^2} H$$
.

Also, from Equation 3, C_{∞} satisfies:

$$HC_{\infty} + C_{\infty}H = \gamma S \circ C_{\infty} + \gamma \Sigma$$
.

Multiplying this by H^{-1} and taking the trace leads to:

$$\operatorname{Tr}(C_{\infty}) = \frac{\gamma}{2} \operatorname{Tr}(H^{-1} \cdot (\mathcal{S} \circ C_{\infty})) + \frac{\gamma}{2} \operatorname{Tr}(H^{-1}\Sigma)$$

$$\leq \frac{1}{2} \frac{\gamma^{2} R^{2}}{1 - \gamma R^{2}} \|\Sigma\|_{H} \operatorname{Tr}(H^{-1}H) + \frac{\gamma}{2} \operatorname{Tr}(H^{-1}\Sigma)$$

$$= \frac{1}{2} \frac{\gamma^{2} R^{2}}{1 - \gamma R^{2}} d\|\Sigma\|_{H} + \frac{\gamma}{2} \operatorname{Tr}(H^{-1}\Sigma)$$

which completes the proof.

3.3 Completing the proof of Theorem 1

Proof. The proof of the theorem is completed by applying the developed lemmas. For the bias term, using convexity leads to:

$$\frac{1}{2}\mathbb{E}[\|\overline{w}_{t:T} - w^*\|_H^2 | \xi_0 = \dots \xi_T = 0] \leq \frac{1}{2}R^2\mathbb{E}[\|\overline{w}_{t:T} - w^*\|^2 | \xi_0 = \dots \xi_T = 0]$$

$$\leq \frac{1}{2}\frac{R^2}{T - t}\sum_{t'=t}^{T-1}\mathbb{E}[\|w_{t'} - w^*\|^2 | \xi_0 = \dots \xi_T = 0]$$

$$\leq \frac{1}{2}\exp(-\gamma \mu t)R^2 \|w_0 - w^*\|^2.$$

For the variance term, observe that

$$\frac{1}{2}\mathbb{E}[\|\overline{w}_{t:T} - w^*\|_H^2|w_0 = w^*] \le \frac{\text{Tr}(C_\infty)}{\gamma(T - t)} \le \frac{1}{T - t} \left(\frac{1}{2}\text{Tr}(H^{-1}\Sigma) + \frac{1}{2}\frac{\gamma R^2}{1 - \gamma R^2}d\|\Sigma\|_H\right),$$

which completes the proof.

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References

- [1] Francis R. Bach. Adaptivity of averaged stochastic gradient descent to local strong convexity for logistic regression. *Journal of Machine Learning Research (JMLR)*, volume 15, 2014.
- [2] Alexandre Défossez and Francis R. Bach. Averaged least-mean-squares: Bias-variance trade-offs and optimal sampling distributions. In *AISTATS*, volume 38, 2015.
- [3] Aymeric Dieuleveut and Francis R. Bach. Non-parametric stochastic approximation with large step sizes. *The Annals of Statistics*, 2015.
- [4] Roy Frostig, Rong Ge, Sham M. Kakade, and Aaron Sidford. Competing with the empirical risk minimizer in a single pass. In *COLT*, 2015.
- [5] Prateek Jain, Sham M. Kakade, Rahul Kidambi, Praneeth Netrapalli, and Aaron Sidford. Parallelizing stochastic approximation through mini-batching and tail-averaging. *CoRR*, abs/1610.03774, 2016.
- [6] Harold J. Kushner and Dean S. Clark. Stochastic Approximation Methods for Constrained and Unconstrained Systems. Springer-Verlag, 1978.
- [7] Erich L. Lehmann and George Casella. Theory of Point Estimation. Springer Texts in Statistics. Springer, 1998.
- [8] Boris T. Polyak and Anatoli B. Juditsky. Acceleration of stochastic approximation by averaging. SIAM Journal on Control and Optimization, volume 30, 1992.
- [9] David Ruppert. Efficient estimations from a slowly convergent robbins-monro process. *Tech. Report*, ORIE, Cornell University, 1988.
- [10] Aad W. van der Vaart. Asymptotic Statistics. Cambridge University Publishers, 2000.