Leverage Score Sampling for Faster Accelerated Regression and ERM

Naman Agarwal; Sham Kakade † Rahul Kidambi † Yin Tat Lee § Praneeth Netrapalli, ¶ Aaron Sidford, ||

November 23, 2017

Abstract

Given a matrix $\mathbf{A} \in \mathbb{R}^{n \times d}$ and a vector $b \in \mathbb{R}^d$, we show how to compute an ϵ -approximate solution to the regression problem $\min_{x \in \mathbb{R}^d} \frac{1}{2} \|\mathbf{A}x - b\|_2^2$ in time $\widetilde{O}((n + \sqrt{d \cdot \kappa_{\text{sum}}}) \cdot s \cdot \log \epsilon^{-1})$ where $\kappa_{\text{sum}} = \text{tr}\left(\mathbf{A}^{\top}\mathbf{A}\right)/\lambda_{\min}(\mathbf{A}^T\mathbf{A})$ and s is the maximum number of non-zero entries in a row of \mathbf{A} . Our algorithm improves upon the previous best running time of $\widetilde{O}((n + \sqrt{n \cdot \kappa_{\text{sum}}}) \cdot s \cdot \log \epsilon^{-1})$.

We achieve our result through a careful combination of leverage score sampling techniques, proximal point methods, and accelerated coordinate descent. Our method not only matches the performance of previous methods, but further improves whenever leverage scores of rows are small (up to polylogarithmic factors). We also provide a non-linear generalization of these results that improves the running time for solving a broader class of ERM problems.

^{*}namana@cs.princeton.edu, Computer Science, Princeton University

[†]sham@cs.washington.edu, University of Washington, Seattle

[‡]rkidambi@uw.edu, University of Washington, Seattle

[§]yintat@uw.edu, University of Washington, Seattle

 $[\]P$ praneeth@microsoft.com, Microsoft Research, India

sidford@stanford.edu, Stanford University

1 Introduction

Given $\mathbf{A} \in \mathbb{R}^{n \times d}$ and $b \in \mathbb{R}^n$, the regression problem $\min_{x \in \mathbb{R}^d} \frac{1}{2} ||\mathbf{A}x - b||_2^2$ is one of the most fundamental problems in optimization and a prominent tool in machine learning. It is one of the simplest empirical risk minimization (ERM) problems and a prominent proving ground for developing efficient learning algorithms.

Regression is long known to be solve-able directly by matrix multiplication in $O(nd^{\omega-1})$ time where $\omega < 2.373$ [Wil12] is the matrix multiplication constant and recent work has improved the running time to $\widetilde{O}((\operatorname{nnz}(\mathbf{A}) + d^{\omega})\log(\epsilon^{-1}))$, i.e. linear time plus the time needed to solve a nearly square linear system. [CW13, LMP13, NN13, CLM⁺15, Coh16] However, for sufficiently large \mathbf{A} even a super-quadratic running time of $\Omega(d^{\omega})$ can be prohibitively expensive. Consequently, over the past decade improving this running time under mild regularity assumptions on \mathbf{A} has been an incredibly active area of research.

In this paper we improve the best known running time for solving regression under standard regularity assumptions. Formally the problem we consider is as follows.

Definition 1 (The Regression Problem). Given $\mathbf{A} \in \mathbb{R}^{n \times d}$ with rows $a_1, ..., a_n$ and $b \in \mathbb{R}^n$, we consider the regression problem $\min_{x \in \mathbb{R}^d} f_{\mathbf{A}, b}(x)$ where

$$f_{\mathbf{A},b}(x) \stackrel{\text{def}}{=} \frac{1}{2} \|\mathbf{A}x - b\|_2^2 = \sum_{i \in [n]} \frac{1}{2} \left(a_i^\top x - b_i \right)^2.$$

The central problem of this paper is to get faster regression algorithms defined as follows.

Definition 2 (Regression Algorithm). We call an algorithm a $\mathcal{T}(\mathbf{A})$ -time regression algorithm if for any $b \in \mathbb{R}^n$, $x_0 \in \mathbb{R}^d$, and $\epsilon \in (0, \frac{1}{2})$ w.h.p in n in time $O(\mathcal{T}(\mathbf{A}) \log \epsilon^{-1})$ the algorithm outputs a vector y such that

$$f_{\mathbf{A},b}(y) - \min_{x} f_{\mathbf{A},b}(x) \le \epsilon \cdot \left(f_{\mathbf{A},b}(x_0) - \min_{x} f_{\mathbf{A},b}(x) \right). \tag{1.1}$$

Note that if x_* is a minimizer of $f_{\mathbf{A},b}(x)$ then the guarantee (1.1) is equivalent to the following

$$||y - x_*||_{\mathbf{A}^\top \mathbf{A}}^2 \le \epsilon ||x_0 - x_*||_{\mathbf{A}^\top \mathbf{A}}^2$$
(1.2)

where $||x||_{\mathbf{M}}^2$ for a PSD matrix M is defined as $x^{\top} \mathbf{M} x$.

The goal of this paper is to provide regression algorithms with improved running times depending on n, d, and the following regularity parameters.

Definition 3. (Regularity Parameters) We let $\lambda_{\min}(\mathbf{A}^{\top}\mathbf{A})$ and $\lambda_{\max}(\mathbf{A}^{\top}\mathbf{A})$ denote the smallest and largest eigenvalues of $\mathbf{A}^{\top}\mathbf{A}$. We let $\kappa(\mathbf{A}^{\top}\mathbf{A}) = \lambda_{\max}(\mathbf{A}^{\top}\mathbf{A})/\lambda_{\min}(\mathbf{A}^{T}\mathbf{A})$ denote the condition number of $\mathbf{A}^{\top}\mathbf{A}$ and let $\kappa_{\text{sum}}(\mathbf{A}^{\top}\mathbf{A}) = \text{tr}(\mathbf{A}^{\top}\mathbf{A})/\lambda_{\min}(\mathbf{A}^{T}\mathbf{A})$ denote the total condition number of $\mathbf{A}^{\top}\mathbf{A}$. We let $s(\mathbf{A})$ denote the maximum number of non-zero entries in a row of \mathbf{A} . Occasionally, we drop the terms in parenthesis when they are clear from context.

¹Through this paper, we use \widetilde{O} to hide factors polylogarithmic in $n,d,\kappa \stackrel{\text{def}}{=} \lambda_{\max}(\mathbf{A}^{\top}\mathbf{A})/\lambda_{\min}(\mathbf{A}^{\top}\mathbf{A})$ and M (c.f. Definition 3 and Definition 6).

1.1 Previous Results

Standard classic iterative methods such as gradient descent and accelerated gradient descent [Nes83] solve the regression problem with running times of $O(n \cdot s(\mathbf{A}) \cdot \kappa(\mathbf{A}^{\top} \mathbf{A}))$ and $O(n \cdot s(\mathbf{A}) \cdot \sqrt{\kappa(\mathbf{A}^{\top} \mathbf{A})})$ respectively. While these running times are super-linear whenever $\kappa(\mathbf{A}^{\top} \mathbf{A})$ is super constant there has been a flurry of recent papers showing that using sampling techniques faster running times can be achieved. These often yield nearly linear running times when n is sufficiently larger than d. [SSZ13, JZ13a, SZ16, All16].

Using recent advances in accelerated coordinate descent [AQRY16, NS17] coupled with proximal point methods [FGKS15a, LMH15] the previous fastest iterative algorithm is as follows:

Theorem 4 (Previous Best Regression Running Times). Given $\mathbf{A} \in \mathbb{R}^{n \times d}$, there is a $\mathcal{T}(\mathbf{A})$ -time regression algorithm with

$$\mathcal{T}(\mathbf{A}) = \widetilde{O}\left(\left(n + \frac{\sum_{i \in [n]} \|a_i\|_2}{\sqrt{\lambda_{\min}(\mathbf{A}^{\top}\mathbf{A})}}\right) \cdot s(\mathbf{A})\right) = \widetilde{O}((n + \sqrt{n \cdot \kappa_{\text{sum}}(\mathbf{A}^{\top}\mathbf{A})}) \cdot s(\mathbf{A})).$$

The inequality in this theorem follows directly from Cauchy Schwartz, as

$$\sum_{i \in [n]} \|a_i\|_2 \le \sqrt{n \cdot \operatorname{tr}(\mathbf{A}^\top \mathbf{A})} = \sqrt{n \cdot \kappa_{\operatorname{sum}}(\mathbf{A}^\top \mathbf{A}) \cdot \lambda_{\min}(\mathbf{A}^\top \mathbf{A})} \ .$$

For the rest of the proof, see Section 5, where we provide a proof of Theorem 13, a generalization of Theorem 4 which is more convenient for our analysis.

1.2 Our Results

The work in this paper is motivated by the natural question, can this running time of Theorem 4 be further improved? Despite the running time lower bound of $\sqrt{n \cdot \kappa_{\text{sum}}(\mathbf{A}^{\top} \mathbf{A})}$ shown in [WS16],² in this paper we give an affirmative answer improving the $\sqrt{n \cdot \kappa_{\text{sum}}(\mathbf{A}^{\top} \mathbf{A})}$ term in Theorem 4 to $\sqrt{d \cdot \kappa_{\text{sum}}(\mathbf{A}^{\top} \mathbf{A})}$. The main result of this paper is the following:

Theorem 5 (Improved Regression Running Time). Given $\mathbf{A} \in \mathbb{R}^{n \times d}$, Algorithm 1 is a $\mathcal{T}(\mathbf{A})$ -time regression algorithm that succeeds with high probability(w.h.p) in n where

$$\mathcal{T}(\mathbf{A}) = \widetilde{O}\left(\operatorname{nnz}(\mathbf{A}) + \left(d + \frac{\sum_{i \in [n]} \|a_i\|_2 \cdot \sigma_i(\mathbf{A})}{\sqrt{\lambda_{\min}(\mathbf{A}^T \mathbf{A})}}\right) \cdot s(\mathbf{A})\right)$$

and $\sigma_i(\mathbf{A}) = ||a_i||_{(\mathbf{A}^\top \mathbf{A})^{-1}}^2$ for all $i \in [n]$.

Up to polylogarithmic factors Theorem 5 is an improvement over Theorem 4 as $\sigma_i(\mathbf{A}) \in [0, 1]$. This improvement can be substantial as $\sigma_i(\mathbf{A})$ can be as small as O(d/n), e.g. if \mathbf{A} is an entry-wise random Gaussian matrix. Compared to Theorem 4 whose second term in running time grows as n, our second term is independent of n due to the following:

$$\sum_{i \in [n]} \|a_i\|_2 \sigma_i(\mathbf{A}) \le \sqrt{\sum_{i \in [n]} \|a_i\|_2^2 \sum_{i \in [n]} \|a_i\|_{(\mathbf{A}^\top \mathbf{A})^{-1}}^2} = \sqrt{\operatorname{tr}(\mathbf{A}^\top \mathbf{A}) \operatorname{tr}(\mathbf{A}(\mathbf{A}^\top \mathbf{A})^{-1} \mathbf{A}^T)} \le \sqrt{d\kappa_{\text{sum}}}.$$
(1.3)

²Their lower bound involves a function with $d \gg n$. However, $d \ll n$ is more common as we explain.

Therefore in Theorem 5 we have $\mathcal{T}(\mathbf{A}) = \widetilde{O}((n + \sqrt{d \cdot \kappa_{\text{sum}}(\mathbf{A}^{\top} \mathbf{A})}) \cdot s(\mathbf{A})).$

This improvement from n to d can be significant as n (the number of samples) is in some cases orders of magnitude larger than d (the number of features). For example, in the LIBSVM dataset³, in 87 out of 106 many non-text problems, we have $n \ge d$, 50 of them have $n \ge d^2$ and in the UCI dataset,⁴ in 279 out of 301 many non-text problems, we have $n \ge d$, 195 out of them have $n \ge d^2$.

Furthermore, in Section 7 we show how to extend our results to ERM problems more general then regression. In particular we consider the following ERM problem

Definition 6 (ERM). Given $\mathbf{A} \in \mathbb{R}^{n \times d}$ with rows $a_1, ..., a_n$ and functions $\{\psi_1 ... \psi_n\} \in \mathbb{R} \to \mathbb{R}$ such that each $\psi_i : \mathbb{R} \to \mathbb{R}$ is twice differentiable and satisfies

$$\forall x \in \mathbb{R}^d \quad \frac{1}{M} \le \psi''(x) \le M \tag{1.4}$$

we wish to minimize $F(x): \mathbb{R}^d \to \mathbb{R}$ over $x \in \mathbb{R}^d$ where

$$F(x) \stackrel{\text{def}}{=} \sum_{i \in [n]} f_i(x) = \sum_{i \in [n]} \psi_i(a_i^T x)$$

While the form of ERM considered by us is not the most general form considered in literature, we consider this problem, to provide a proof of concept that our techniques for regression are more broadly applicable to the case of ERM and can if fact provide a non-trivial improvement over existing methods. We leave it to future work to study the limitations of our techniques and possible improvements for more general ERM problems. The following is our main theorem regarding the ERM problem.

Theorem 7. Given an ERM problem(Definition 6) and an initial point x_0 , there exists an algorithm that produces a point x' such that $F(x') - F_{x_*} \leq \epsilon (F(x_0) - F_{x_*})$ which succeeds w.h.p in n in total time

$$\tilde{O}\left(\left(\operatorname{nnz}(\mathbf{A}) + \left(dM^5 + \sum_{i=1}^n \frac{\|a_i\|_2 \sqrt{\sigma_i} M^3}{\sqrt{\lambda_{\min}(\mathbf{A}^\top \mathbf{A})}}\right) s(\mathbf{A})\right) \log\left(\frac{1}{\epsilon}\right)\right)$$

where $\sigma_i \stackrel{\text{def}}{=} a_i^T [\mathbf{A}^\top \mathbf{A}]^{-1} a_i$ are the leverage scores with respect to a_i .

Note that Theorem 7 interpolates our regression results, i.e. it recovers our results for regression in the special case of M=1. To better understand the bound in Theorem 7, note that following the derivation in Equation (1.3) we have that the running time in Theorem 7 is bounded by

$$\tilde{O}\left(\left(\operatorname{nnz}(\mathbf{A}) + \left(dM^5 + \sum_{i=1}^n M^3 \sqrt{d\kappa_{\operatorname{sum}}(\mathbf{A}^\top \mathbf{A})}\right) s(\mathbf{A})\right) \log\left(\frac{1}{\epsilon}\right)\right) .$$

The best known bound for the ERM problem as defined in Definition 6 given by [FGKS15a, LMH15, All16] is

$$\tilde{O}\left(\left(\operatorname{nnz}(\mathbf{A}) + \sum_{i=1}^{n} M \sqrt{n\kappa_{\operatorname{sum}}(\mathbf{A}^{\top}\mathbf{A})}\right) s(\mathbf{A}) \log\left(\frac{1}{\epsilon}\right)\right)$$

³https://www.csie.ntu.edu.tw/~cjlin/libsvmtools/datasets/

⁴http://archive.ics.uci.edu/ml/datasets.html

In this case Theorem 7 should be seen as implying that under Assumption (1.4) the effective dependence on the number of examples on the running time for ERM can be reduced to at most dM^5 .

Again, we remark that the running time bound of Theorem 7 should be viewed as a proof of concept that our regression machinery can be used to improve the running time of ERM. We leave it as future work to both improve Theorem 7's dependence on M and have it extend to a broader set of problems. For example, we believe the trunning time can be immediately improved to

$$\tilde{O}\left(\left(\operatorname{nnz}(\mathbf{A}) + \left(dM^4 + \sum_{i=1}^n \frac{\|a_i\|_2 \sqrt{\sigma_i} M^3}{\sqrt{\lambda_{\min}(\mathbf{A}^\top \mathbf{A})}}\right) s(\mathbf{A})\right) \log\left(\frac{1}{\epsilon}\right)\right)$$

simply by using a proximal version of Theorem 18, which is alluded to in the work of [AQRY16]. Note that this improvement leads to the effective number of examples being bounded by dM^4 .

1.3 Our Approach

Our algorithm follows from careful combination and analysis of a recent suite of advances in numerical linear algebra. First, we use the previous fastest regression algorithm, Theorem 4, which is the combination of recent advances in accelerated coordinate descent [AQRY16, NS17] and proximal point methods [FGKS15a, LMH15] (See Section 5.) Then, we show that if we have estimates of the leverage scores of the rows of **A**, a natural recently popularized measure of importance, [SS08, LMP13, CLM⁺15] we can use concentration results on leverage score sampling and preconditioning to obtain a faster regression algorithm. (See Section 3.)

Now, it is a powerful and somewhat well known fact that given an algorithm for regression one can compute leverage score estimates in nearly linear time plus the time needed to solve $\widetilde{O}(1)$ regression problems [SS08]. Consequently, to achieve the improved running time when we do not have leverage scores we are left with a chicken and egg problem. Fortunately, recent work has shown that such a problem can be solved in a several ways [LMP13, CLM⁺15]. We show that the technique in [LMP13] carefully applied and analyzed can be used to obtain our improved running time for both estimating leverage scores and solving regression problems with little overhead. (See Section 4.)

To generalize our results for a broader class of ERM problems we follow a similar procedure. Most parts generalize naturally, however perhaps the most complex ingredient is how to generalize preconditioning to the case when we are sampling non-quadratic functions. For this, we prove concentration results on sampling from ERM inspired from [FGKS15b] to show that it suffices to solve ERM on a sub-sampling of the components that may be of intrinsic interest. (See Section 7)

In summary our algorithms are essentially a careful blend of accelerated coordinate descent and concentration results coupled with the iterative procedure in [LMP13] and the Johnson Lindenstrauss machinery of [SS08] to compute leverage scores. Ultimately the algorithms we provide are fairly straightforward, but it provides a substantial running time improvement that we think is of intrinsic interest. We hope this work may serve as a springboard for even faster iterative methods for regression and minimizing finite sums more broadly.

Finally, we remark that there is another way to achieve the $\sqrt{d \cdot \kappa_{\text{sum}}(\mathbf{A})}$ improvement over $\sqrt{n \cdot \kappa_{\text{sum}}(\mathbf{A})}$. One could simply use subspace embeddings [CW13, NN13, Coh16] and preconditioning to reduce the regression problem to a regression problem on a $\widetilde{O}(d) \times d$ matrix and then apply Theorem 4 to solve the $\widetilde{O}(d) \times d$ regression problem. While this works, it has three shortcomings

relevant to our approach. First, this procedure would possibly lose sparsity, the $\widetilde{O}(d) \times d$ matrix may be dense and thus the final running time would have an additional $\widetilde{O}(d^2)$ term our method does not. Second, it is unclear if this approach yields our more fine-grained running time dependence on leverage scores that appears in Theorem 5 which we believe to be significant. Thirdly it is unclear how to extend the approach to the ERM setting.

1.4 Paper Organization

After providing requisite notation in Section 2 we prove Theorem 5 in Sections 3 and 4. We first provide the algorithm for regression given leverage score estimates in Section 3 and further provide the algorithm to compute the estimates based in Section 4. Note that the algorithm for computing leverage scores makes use of the algorithm for regression given leverage scores as a sub-routine. In Section 5 we provide the proofs of deferred lemmas and theorems from Sections 3, 4. In Section 7 we provide the proof of Theorem 7. Further we collect some simple re-derivations and reductions from known results required through the paper in the Appendix.

2 Notation

For symmetric matrix $\mathbf{M} \in \mathbb{R}^{d \times d}$ and $x \in \mathbb{R}^d$ we let $||x||_{\mathbf{M}}^2 = x^{\top} \mathbf{M} x$. For symmetric matrix $\mathbf{N} \in \mathbb{R}^{d \times d}$ we use $\mathbf{M} \leq \mathbf{N}$ to denote the condition that $x^{\top} \mathbf{M} x \leq x^{\top} \mathbf{N} x$ for all $x \in \mathbb{R}^d$ and we define \prec , \succeq , and \succ analogously. We use $\mathrm{nnz}(\mathbf{A})$ to denote the number of non-zero entries in \mathbf{A} and for a vector $b \in \mathbb{R}^n$ we let $\mathrm{nnz}(b)$ denote the number of nonzero entries in b.

3 Regression Algorithm Given Leverage Score Estimates

The regression algorithm we provide in this paper involves two steps. First we find which rows of **A** are important, where importance is measured in terms of *leverage score*. Second, we use these leverage scores to sample the matrix and solve the regression problem on the sampled matrix using Theorem 13. In this section we introduce leverage scores and provide and analyze the second step of our algorithm.

Definition 8 (Leverage Score). For $\mathbf{A} \in \mathbb{R}^{n \times d}$ with rows $a_1, ..., a_n \in \mathbb{R}^d$ we denote the *leverage* score of row $i \in [n]$ by $\sigma_i(\mathbf{A}) \stackrel{\text{def}}{=} a_i^\top (\mathbf{A}^\top \mathbf{A})^+ a_i$.

Leverage score have numerous applications and are well studied. It is well known that $\sigma_i(\mathbf{A}) \in (0,1]$ for all $i \in [n]$ and $\sum_{i \in [n]} \sigma_i(\mathbf{A}) = \operatorname{rank}(\mathbf{A})$. The critical fact we used about leverage scores is that sampling rows of \mathbf{A} according to any overestimate of leverage scores yields a good approximation to \mathbf{A} after appropriate re-scaling [CLM⁺15, SS08]:

Lemma 9 (Leverage Score Sampling (Lemma 4 of [CLM+15])). Let $\mathbf{A} \in \mathbb{R}^{n \times d}$, let $\delta \in (0, \frac{1}{2})$, and let $u \in \mathbb{R}^n$ be overestimates of the leverage scores of \mathbf{A} ; i.e. $u_i \geq \sigma_i(\mathbf{A})$ for all $i \in [n]$. Define $p_i \stackrel{\text{def}}{=} \min \{1, k\delta^{-2}u_i \log n\}$ for a sufficiently large absolute constant k > 0 and let $\mathbf{H} \in \mathbb{R}^{n \times n}$ be a random diagonal matrix where independently $\mathbf{H}_{ii} = \frac{1}{p_i}$ with probability p_i and $\mathbf{H}_{ii} = 0$ otherwise. With high probability in n, $\operatorname{nnz}(\mathbf{H}) = O(d \cdot \delta^{-2} \cdot \log n)$ and $(1 - \delta) \cdot \mathbf{A}^{\top} \mathbf{A} \preceq \mathbf{A}^{\top} \mathbf{H} \mathbf{A} \preceq (1 + \delta) \cdot \mathbf{A}^{\top} \mathbf{A}$.

$\overline{\mathbf{Algorithm}}$ 1: SolveUsingLS $_{\mathbf{A},u}(x_0,b,\epsilon)$

Let $p_i = \min\{1, k' \cdot u_i \log n\}$ where k' is a sufficiently large absolute constant.

repeat

Let $\mathbf{H} \in \mathbb{R}^{n \times n}$ be a diagonal matrix where independently for all $i \in [n]$ we let $\mathbf{H}_{ii} = \frac{1}{n_i}$ with probability p_i and 0 otherwise.

Let $\mathbf{B} = \sqrt{\mathbf{H}} \mathbf{A}$.

until $\sum_{i \in [n]} ||b_i||_2 \le 2 \cdot \sum_{i \in [n]} \sqrt{k' \cdot u_i \log n} \cdot ||a_i||_2$; Invoke Theorem 13 on **A** and **B** to find y such that

$$f_{\mathbf{A},b}(y) - \min_{x} f_{\mathbf{A},b}(x) \le \epsilon \cdot \left(f_{\mathbf{A},b}(x_0) - \min_{x} f_{\mathbf{A},b}(x) \right).$$

Output: y.

Theorem 10. If $u \in \mathbb{R}^n$ satisfies $\sigma_i(\mathbf{A}) \leq u_i \leq 4 \cdot \sigma_i(\mathbf{A}) + [n \cdot \kappa(\mathbf{A}^\top \mathbf{A})]^{-1}$ for all $i \in [n]$ then ${ t SolveUsingLS_{{f A},u}}$ is a ${\cal T}({f A})$ -time regression algorithm where

$$\mathcal{T}(\mathbf{A}) = \widetilde{O}\left(\operatorname{nnz}(\mathbf{A}) + \left(d + \frac{\sum_{i \in [n]} \sqrt{\sigma_i(\mathbf{A})} \cdot ||a_i||_2}{\sqrt{\lambda_{\min}(\mathbf{A}^\top \mathbf{A})}}\right) \cdot s(\mathbf{A})\right).$$

Proof. Let $k' = \delta^{-2} \cdot k$ for $\delta = \frac{1}{10}$. Applying Lemma 9 yields with high probability in n that

$$\left(\frac{5}{6}\right) \mathbf{A}^{\top} \mathbf{A} \leq \mathbf{A}^{\top} \mathbf{H} \mathbf{A} \leq \left(\frac{6}{5}\right) \mathbf{A}^{\top} \mathbf{A} \tag{3.1}$$

where $\mathbf{A}^{\top}\mathbf{H}\mathbf{A} = \sum_{i \in [n]: \mathbf{H}_{ii} \neq 0} b_i b_i^{\top}$, $b_i \stackrel{\text{def}}{=} \frac{1}{\sqrt{p_i}} a_i$ and $p_i \stackrel{\text{def}}{=} \min\{1, k' \cdot u_i \log n\}$. Note that

$$\mathbb{E}\left[\sum_{i\in[n]} \|b_i\|_2\right] = \sum_{i\in[n]} \frac{p_i}{\sqrt{p_i}} \|a_i\|_2 \le \sum_{i\in[n]} \sqrt{k'u_i \log n} \|a_i\|_2.$$

Consequently, by Markov's inequality

$$\sum_{i \in [n]} \|b_i\|_2 \le 2 \cdot \sum_{i \in [n]} \sqrt{k' \cdot u_i \log n} \cdot \|a_i\|_2$$

with probability at least 1/2 and the loop in the algorithm terminates with high probability in n in $O(\log n)$ iterations. Consequently, the loop takes only $O(\operatorname{nnz}(\mathbf{A}) + n \log n)$ -time and since we only sampled $O(\log n)$ many independent copies of $\mathbf{A}^{\top}\mathbf{H}\mathbf{A}$, the guarantee (3.1) again holds with high probability in n.

Using the guarantee (3.1) and Theorem 13 on **A** and $\mathbf{B} \stackrel{\text{def}}{=} \sqrt{\mathbf{H}}\mathbf{A}$, we can produce a y we need in time $O(\log(\epsilon^{-1}))$ times

$$\widetilde{O}\left(\left(\operatorname{nnz}(\mathbf{A}) + \left(d\log n + \frac{1}{\sqrt{\lambda}}\sum_{i\in[n]}\|b_i\|_2\right) \cdot s(\mathbf{A})\right)\right)$$

where we used that **B** has at most $O(d \log n)$ rows with high probability in n. Since we know

$$\sum_{i \in [n]} \|b_i\|_2 \le 2 \sum_{i \in [n]} \sqrt{k' \cdot u_i \log n} \cdot \|a_i\|_2,$$

all that remains is to bound $\sum_{i \in [n]} \sqrt{u_i} \|a_i\|_2$. However, $\mathbf{A}^{\top} \mathbf{A} \leq \lambda_{\max}(\mathbf{A}^{\top} \mathbf{A}) \mathbf{I}$ and therefore

$$\mathbf{I} \leq \lambda_{\max}(\mathbf{A}^{\top}\mathbf{A})(\mathbf{A}^{\top}\mathbf{A})^{-1} \text{ and } \|a_i\|_2 \leq \sqrt{\lambda_{\max}(\mathbf{A}^{\top}\mathbf{A})} \cdot \sigma_i(\mathbf{A}) .$$

Consequently, Cauchy Schwartz and $\lambda_{\min}(\mathbf{A}^{\top}\mathbf{A}) \leq \operatorname{tr}(\mathbf{A}^{\top}\mathbf{A})$ yields

$$\frac{1}{\sqrt{n}} \sum_{i \in [n]} \|a_i\|_2 \le \sqrt{\sum_{i \in [n]} \|a_i\|_2^2} \le \frac{1}{\sqrt{\lambda_{\min}(\mathbf{A}^\top \mathbf{A})}} \sum_{i \in [n]} \|a_i\|_2^2 \le \sqrt{\kappa(\mathbf{A}^\top \mathbf{A})} \sum_{i \in [n]} \sqrt{\sigma_i(\mathbf{A})} \cdot \|a_i\|_2.$$

Since $\sqrt{a+b} \le \sqrt{a} + \sqrt{b}$ this yields

$$\sum_{i \in [n]} \sqrt{u_i} \cdot \|a_i\|_2 \le 2 \sum_{i \in [n]} \sqrt{\sigma_i(\mathbf{A})} \cdot \|a_i\|_2 + \frac{1}{\sqrt{n \cdot \kappa(\mathbf{A}^\top \mathbf{A})}} \sum_{i \in [n]} \|a_i\|_2 \le 3 \sum_{i \in [n]} \sqrt{\sigma_i(\mathbf{A})} \cdot \|a_i\|_2$$

which in turn yields the result as \widetilde{O} hides factors poly-logarithmic in n and d.

4 Regression Algorithm Without Leverage Score Estimates

In the previous section we showed that we can solve regression in our desired running time provided we have constant factor approximation to leverage scores. Here we show how to apply this procedure repeatedly to estimate leverage scores as well. We do this by first adding a large multiple of the identity to our matrix and then gradually decreasing this multiple while maintaining estimates for leverage scores along the way. This is a technique introduced in [LMP13] and we leverage it tailored to our setting.

A key technical ingredient for this algorithm is the following well-known result on the reduction from leverage score computation to regression with little overhead. Formally, Lemma 11 states that you can compute constant multiplicative approximations to all leverage scores of a matrix in nearly linear time plus the time needed to solve $\widetilde{O}(1)$ regression problems. Algorithm 2 details the procedure for computing leverage scores.

Lemma 11 (Computing Leverage Scores). For $\mathbf{A} \in \mathbb{R}^{n \times d}$ let \mathcal{A} be a $\mathcal{T}(\mathbf{A})$ -time algorithm for regression on \mathbf{A} . For $\delta \in (\frac{1}{n}, \frac{1}{2})$, in time $O((\operatorname{nnz}(\mathbf{A}) + \mathcal{T}(\mathbf{A}) \log \epsilon^{-1}) \delta^{-2} \log n)$ where we set $\epsilon = \delta^2 (18n \cdot d \cdot \log n \cdot \kappa(\mathbf{A}^{\top}\mathbf{A}))^{-2}$, with high probability in n, the algorithm ComputeLS($\mathbf{A}, \delta, \mathcal{A}$) outputs $\tau \in \mathbb{R}^n$ such that $\sigma_i(\mathbf{A}) \leq \tau_i \leq (1 + \delta)\sigma_i(\mathbf{A}) + \delta \cdot [n \cdot \kappa(\mathbf{A}^{\top}\mathbf{A})]^{-1}$. for all $i \in [n]$.

We defer the proof of Lemma 11 to the Appendix (Section A). Combining the algorithm for estimating leverage scores ComputeLS (Algorithm 2) with our regression algorithm given leverage scores SolveUsingLS (Theorem 10) yields our solver (Algorithm 3). We first provide a technical lemma regarding invariants maintained by the algorithm, Lemma 12, and then we prove that this algorithm has the desired properties to prove Theorem 5.

Algorithm 2: ComputeLS($\mathbf{A}, \delta, \mathcal{A}$)

Let $k = c \log(n)$ and $\epsilon = \frac{\delta^2}{(18nd \log n \cdot \kappa(\mathbf{A}^{\top} \mathbf{A}))^2}$ where c is some large enough constant.

for $j = 1, \dots, k$ do

Let $v_i \in \mathbb{R}^n$ be a random Gaussian vector, i.e. each entry follows N(0, I). Use algorithm \mathcal{A} to find a vector y_i such that

$$f_{\mathbf{A},v_j}(y_j) - \min_x f_{\mathbf{A},v_j}(x) \le \epsilon (f_{\mathbf{A},v_j}(0) - \min_x f_{\mathbf{A},v_j}(x))$$
.

end

Let $\tau_i = \frac{1}{k} \sum_{j=1}^k (e_i^{\top} \mathbf{A}^{\top} y_j)^2$ for all $i = 1, \dots, n$. **Output:** $\frac{\tau}{1 - \delta/3} + \frac{\delta}{2n \cdot \kappa(\mathbf{A}^{\top} \mathbf{A})}$.

Algorithm 3: Solve_A (x_0, b, ϵ)

Let
$$\mathbf{A}_{\eta} = \begin{pmatrix} \mathbf{A} \\ \sqrt{\eta} \mathbf{I} \end{pmatrix}$$
, $\eta = \lambda_{\max}(\mathbf{A}^{\top} \mathbf{A})$ and $u_i = \begin{cases} \frac{1}{\eta} ||a_i||_2^2 & \text{if } 1 \leq i \leq n \\ 1 & \text{if } n+1 \leq i \leq n+d \end{cases}$.

Set $\eta \leftarrow 0$. Let $\overline{b} = \begin{pmatrix} b \\ \vec{0} \end{pmatrix} \in \mathbb{R}^{n+d}$.

Apply algorithm SolveUsingLS $_{\mathbf{A}_0,u}$ to find y such that

$$f_{\mathbf{A}_0,\overline{b}}(y) - \min_x f_{\mathbf{A}_0,\overline{b}}(x) \le \epsilon (f_{\mathbf{A}_0,\overline{b}}(x_0) - \min_x f_{\mathbf{A}_0,\overline{b}}(x)) \ .$$

Output: y

Lemma 12. In the algorithm Solve_{A, ϵ} (See Algorithm 3) the following invariant is satisfied

$$\sigma_i(\mathbf{A}_{\eta}) \le u_i \le 4 \cdot \sigma_i(\mathbf{A}_{\eta}) + [n \cdot \kappa(\mathbf{A}_{\eta}^{\top} \mathbf{A}_{\eta})]^{-1}.$$
(4.1)

Proof. Note that $\mathbf{A}_{\eta}^{\top}\mathbf{A}_{\eta} = \mathbf{A}^{\top}\mathbf{A} + \eta \mathbf{I}$. Consequently, since initially $\eta = \lambda_{\max}(\mathbf{A}^{\top}\mathbf{A})$ we have that initially $\eta \mathbf{I} \leq \mathbf{A}_{\eta}^{\top} \mathbf{A}_{\eta} \leq 2\eta \mathbf{I}$. Consequently, we have that initially $\sigma_i(\mathbf{A}_{\eta}) \leq u_i \leq 2\sigma_i(\mathbf{A}_{\eta})$ and therefore satisfies the invariant (4.1).

Now, suppose at the start of the repeat loop, u satisfies the invariant (4.1). In this case the the assumptions needed to invoke SolveUsingLS by Theorem 10 are satisfied. Hence, after the line $u \leftarrow 2 \cdot \texttt{ComputeLS}(\mathbf{A}_{\eta}, \frac{1}{4}, \mathcal{A}), \text{ by Lemma 11 we have that for all } i \in [n]$

$$2\sigma_i(\mathbf{A}_{\eta}) \le u_i \le 2\left(1 + \frac{1}{4}\right)\sigma_i(\mathbf{A}_{\eta}) + \frac{2}{4n \cdot \kappa(\mathbf{A}_{\eta}^{\top}\mathbf{A}_{\eta})}.$$

Now, letting $\eta' = \frac{3}{4}\eta$ we see that $(3/4)\sigma_i(\mathbf{A}_{\eta}) \leq \sigma_i(\mathbf{A}_{\eta'}) \leq (4/3)\sigma_i(\mathbf{A}_{\eta})$ and direct calculation shows that invariant (4.1) is still satisfied after changing η to η' .

All the remains is to consider the last step when we set $\eta = 0$. When this happens $\eta < \frac{1}{10}\lambda_{\min}(\mathbf{A}^{\top}\mathbf{A})$. and therefore $\sigma_i(\mathbf{A}_{\eta})$ is close enough to $\sigma_i(\mathbf{A})$ and the invariant (4.1) is satisfied. \square

Using this we prove that our main algorithm works as desired.

Proof of Theorem 5. Lemma 12 shows that u is always a good estimate enough of $\sigma_i(\mathbf{A}_{\eta})$ throughout the algorithm to invoke SolveUsingLS with Theorem 10. In particular, this holds at the last step when η is set to 0 and thus the output of the algorithm is as desired by Theorem 10.

During the whole algorithm, ComputeLS($\mathbf{A}_{\eta}, \frac{1}{4}, \mathcal{A}$) is called $\Theta(\log(\kappa(\mathbf{A}^{\top}\mathbf{A})))$ times. Each time ComputeLS is called, SolveUsingLS is called $\Theta(\log(n))$ many times. All that remains is to bound the running time of SolveUsingLS. However, for $\lambda \geq 0$ and $i \in [n]$ we have $\sigma_i(\mathbf{A}_{\lambda}) \leq \sigma_i(\mathbf{A}_0)$ and since $\mathbf{A}_{\lambda}^{\top}\mathbf{A}_{\lambda} \geq \lambda \mathbf{I}$ we $\lambda \leq \lambda_{\min}(\mathbf{A}_{\lambda}^{\top}\mathbf{A}_{\lambda})$. Furthermore, since $\lambda_{\min}(\mathbf{A}_{\lambda}^{\top}\mathbf{A}_{\lambda}) \geq \lambda_{\min}(\mathbf{A}^{\top}\mathbf{A})$ we have that the running time follows from the following:

$$\frac{\sum_{i \in [n+d]} \sqrt{\sigma_i(\mathbf{A}_{\lambda})} \cdot \|a_i\|_2}{\sqrt{\lambda_{\min}(\mathbf{A}_{\lambda}^{\top} \mathbf{A}_{\lambda})}} \leq \sum_{i \in [n]} \frac{\sqrt{\sigma_i(\mathbf{A})} \cdot \|a_i\|_2}{\sqrt{\lambda_{\min}(\mathbf{A}^{\top} \mathbf{A})}} + \sum_{i \in [d]} \frac{\sqrt{\lambda}}{\sqrt{\lambda}}.$$

5 Previous Best Running Time for Regression

Here we state Theorem 13, a generalization of Theorem 4 that is useful for our analysis. Theorem 13 follows by applying recent results on accelerated coordinate descent [AQRY16, NS17] to the dual of regression through recent results on approximate proximal point / Catalyst [FGKS15a, LMH15].

Theorem 13 (Previous Best Regression Running Time). Let **A** and **B** be matrices with the same number of columns. Suppose that **B** has n rows and $(\frac{5}{6})$ $\mathbf{B}^{\top}\mathbf{B} \leq \mathbf{A}^{\top}\mathbf{A} \leq (\frac{6}{5})$ $\mathbf{B}^{\top}\mathbf{B}$, then there is a $\mathcal{T}(\mathbf{A})$ -time regression algorithm with

$$\mathcal{T}(\mathbf{A}) = \widetilde{O}\left(\operatorname{nnz}(\mathbf{A}) + \left(n + \frac{\sum_{i \in [n]} \|b_i\|_2}{\sqrt{\lambda_{\min}(\mathbf{B}^{\top}\mathbf{B})}}\right) \cdot s(\mathbf{B})\right).$$

5.1 Proof of Theorem 13

First we give the theorems encapsulating the results we use and then use them to prove Theorem 4 in the case when $\mathbf{A} = \mathbf{B}$. We then prove the case when $\mathbf{A} \neq \mathbf{B}$. Theorem 14 describes the fastest coordinate descent algorithm known by [AQRY16]. Theorem 15 describes the reduction [FGKS15a] to from regression to coordinate decent via proximal point.

Theorem 14 (Corollary of Thm 5.1 of [AQRY16]). Let $f: \mathbb{R}^n \to \mathbb{R}$ be a twice differentiable σ strongly convex function for $\mu > 0$. Further suppose that for all $x \in \mathbb{R}^n$ and $i \in [n]$ it is the case
that $\frac{\partial^2}{\partial x_i^2} f(x) \leq L_i$ for $i \in [n]$ and the partial derivative $\frac{\partial}{\partial x_i} f(x)$ can be computed in O(s) time.
Then there exists an algorithm which given any $\epsilon > 0$ finds a $y \in \mathbb{R}^n$ such that

$$f(y) - \min_{x} f(x) \le \epsilon \left(f(x_0) - \min_{x} f(x) \right).$$

in expected running time $O(s \sum_i \sqrt{L_i/\mu})$.

Theorem 15 (Corollary of Thm 4.3 of [FGKS15a]). Given $\mathbf{A} \in \mathbb{R}^{n \times d}$ with rows $a_1, ..., a_n$ and $c \in \mathbb{R}^n$. Consider the function $p(x) = \sum_{i=1}^n \phi_i(a_i^\top x)$ where ϕ_i are convex functions. Suppose that $\lambda \mathbf{I} \leq \nabla^2 p(x) \leq L\mathbf{I}$ for all $x \in \mathbb{R}^d$. Let $\kappa = L/\lambda$. Let dual problem $g_s(y) = \sum_{i=1}^n \phi_i^*(y_i) + \frac{1}{2\lambda} \|\mathbf{A}^\top y\|_2^2 - s^\top \mathbf{A}^\top y$.

Suppose that for any $s \in \mathbb{R}^d$, any $y_0 \in \mathbb{R}^n$ and any $0 \le \epsilon \le \frac{1}{2}$, we can compute y in expected running time \mathcal{T}_{ϵ} such that

$$g_s(y) - \min_{y} g_s(y) \le \epsilon (g_s(y_0) - \min_{y} g_s(y)).$$
 (5.1)

Then, for any x_0 and any $\epsilon \in (0, \frac{1}{2})$ we can find x such that

$$p(x) - \min_{x} p(x) \le \epsilon \left(p(x_0) - \min_{x} p(x) \right)$$

in time $\tilde{O}(\mathcal{T}_{\delta}\log(1/\epsilon))$ w.h.p. in n where $\delta = \Theta(n^{-2}\kappa^{-4})$ and \tilde{O} includes logarithmic factors in n, κ .

We note that although the guarantees of Thm 5.1 of [AQRY16] and Thm 4.3 of [FGKS15a] are not stated in the form of Theorems 14 and 15. They can be easily converted to the form above by noticing that the expected running time of the procedure in Thm 4.3 of [FGKS15a] using Theorem 14 is $\tilde{O}(T_{\delta} \log(1/\epsilon))$ which can then be boosted to high probability in n using Lemma 16. We now give the proof of Theorem 13.

Proof of Theorem 13 when $\mathbf{A} = \mathbf{B}$. Let

$$p(x) = \sum_{i=1}^{n} \phi_i(a_i^{\top} x) \text{ where } \phi_i(x) = \frac{1}{2} (x - b_i)^2.$$

Then, we have that $\phi_i^*(y) = \frac{1}{2}y^2 + b_i y$ and hence

$$g_s(y) = \sum_{i=1}^n \phi_i^*(y_i) + \frac{1}{2\lambda} \|\mathbf{A}^\top y\|_2^2 - s^\top \mathbf{A}^\top y = \frac{1}{2} \|y\|_2^2 + b^\top y + \frac{1}{2\lambda} \|\mathbf{A}^\top y\|_2^2 - s^T \mathbf{A}^T y.$$

Note that $g_s(y)$ is 1 strongly convex and

$$\frac{d^2}{dy_i^2} g_s(y) = 1 + \frac{1}{\lambda} ||a_i||_2^2 \stackrel{\text{def}}{=} L_i .$$

Hence, Theorem 14 finds y satisfying (5.1) in time

$$O\left(s(\mathbf{A}) \cdot \sum_{i \in [n]} \sqrt{1 + \frac{1}{\lambda} \|a_i\|_2^2} \log(\epsilon^{-1})\right) = O\left(\left(n + \frac{1}{\sqrt{\lambda}} \sum_{i \in [n]} \|a_i\|_2\right) \cdot s(\mathbf{A}) \cdot \log(\epsilon^{-1})\right).$$

Hence, this shows that the primal can be solved in time

$$O\left(\left(n + \frac{1}{\sqrt{\lambda}} \sum_{i \in [n]} \|b_i\|_2\right) \cdot s \cdot \log(n \cdot \kappa) \cdot \log(\kappa \epsilon^{-1})\right)$$

where we used $\mathbf{A} = \mathbf{B}$ at the end.

Proof of Theorem 13 for the case $\mathbf{A} \neq \mathbf{B}$. The proof involves two steps. First, we show that given any point x_0 , we can find a new point x that is closer to the minimizer. Then, we bound how many steps it takes. To find x, we consider the function

$$f_{x_0}(x) = \frac{1}{2} \|\mathbf{B}x - \mathbf{B}x_0\|_2^2 + \langle \mathbf{A}x_0 - c, \mathbf{A}x - \mathbf{A}x_0 \rangle.$$

Let z be the minimizer of f_{x_0} and x^* be the minimizer of $\frac{1}{2}\|\mathbf{A}x - b\|_2^2$. Note that

$$z = x_0 - (\mathbf{B}^{\mathsf{T}} \mathbf{B})^{-1} \mathbf{A}^{\mathsf{T}} \eta \text{ with } \eta = \mathbf{A} x_0 - b, \text{ and } x^* = (\mathbf{A}^{\mathsf{T}} \mathbf{A})^{-1} \mathbf{A}^{\mathsf{T}} b.$$

Hence, we have that

$$\frac{1}{2} \|\mathbf{A}z - \mathbf{A}x^*\|_2^2 = \frac{1}{2} \|\mathbf{A}(\mathbf{A}^\top \mathbf{A})^{-1} \mathbf{A}^\top \eta - \mathbf{A}(\mathbf{B}^\top \mathbf{B})^{-1} \mathbf{A}^\top \eta\|_2^2
= \frac{1}{2} \eta \mathbf{A}^\top (\mathbf{A}^\top \mathbf{A})^{-1} \mathbf{A}^\top \eta - \eta \mathbf{A}^\top (\mathbf{B}^\top \mathbf{B})^{-1} \mathbf{A}^\top \eta + \frac{1}{2} \|\mathbf{A}(\mathbf{B}^\top \mathbf{B})^{-1} \mathbf{A}^\top \eta\|_2^2.$$

Using that $\frac{5}{6}\mathbf{B}^{\top}\mathbf{B} \leq \mathbf{A}^{\top}\mathbf{A} \leq \frac{6}{5}\mathbf{B}^{\top}\mathbf{B}$, we have

$$\frac{1}{2} \|\mathbf{A}z - \mathbf{A}x^*\|_2^2 \le \frac{4}{10} \eta \mathbf{A}^\top (\mathbf{A}^\top \mathbf{A})^{-1} \mathbf{A}^\top \eta = \frac{4}{10} \|\mathbf{A}x_0 - \mathbf{A}x^*\|_2^2.$$
 (5.2)

However, it is difficult to reduce to the case when $\mathbf{A} = \mathbf{B}$ to minimize the function f_{x_0} due to the extra linear term. To address this issue, we assume $\mathbf{B} = [\overline{\mathbf{B}}; \sqrt{\frac{\lambda}{100}}\mathbf{I}]$ by appending an extra identity term. Note that this only adds a small matrix $\frac{\lambda}{100}\mathbf{I}$ and hence we still have $\frac{5}{6}\mathbf{B}^{\top}\mathbf{B} \leq \mathbf{A}^{\top}\mathbf{A} \leq \frac{6}{5}\mathbf{B}^{\top}\mathbf{B}$ but with a slightly different constant which will not affect the proof for (5.2). Due to the extra identity term, $f_{x_0}(x)$ reduces to an expression of the form $\frac{1}{2}||\mathbf{B}x - d||_2^2 + C$ for some vector d and constant C. We can now apply Theorem 13 for the case $\mathbf{A} = \mathbf{B}$ and get an x such that

$$f_{x_0}(x) - f_{x_0}(z) \le \frac{1}{200} \left(f_{x_0}(x_0) - f_{x_0}(z) \right).$$
 (5.3)

in time

$$O\left(\left(n + \frac{\sum_{i \in [n]} \|b_i\|_2}{\sqrt{\lambda_{\min}(\mathbf{B}^{\top}\mathbf{B})}}\right) \cdot s(\mathbf{B}) \cdot \log(n\kappa) \cdot \log(\kappa)\right) .$$

Note that the extra terms in **B** does not affect the minimum eigenvalue and it increases $\frac{1}{\sqrt{\lambda}} \sum_{i \in [n]} ||b_i||_2$ by at most n.

Now, using the formula of z, the guarantee (5.3) can be written as

$$\|\mathbf{B}x - \mathbf{B}z\|_2^2 \le \frac{1}{200} \|\mathbf{A}x_0 - \mathbf{A}x^*\|_2^2$$
.

Using that $\frac{5}{6}\mathbf{B}^{\top}\mathbf{B} \leq \mathbf{A}^{\top}\mathbf{A} \leq \frac{6}{5}\mathbf{B}^{\top}\mathbf{B}$, we have

$$\|\mathbf{A}x - \mathbf{A}z\|_2 \le \frac{1}{10} \|\mathbf{A}x_0 - \mathbf{A}x^*\|_2$$
.

Combining this with (5.2), we have that

$$\|\mathbf{A}x - \mathbf{A}x^*\|_2 \le 0.9 \|\mathbf{A}x_0 - \mathbf{A}x^*\|_2$$
.

Hence, we get closer to x^* by constant factor. Therefore, to achieve (1.2), we only need to repeat this process $\log(1/\epsilon)$ times. Hence, the total running time is

$$O\left(\left(n + \frac{\sum_{i \in [n]} \|b_i\|_2}{\sqrt{\lambda_{\min}(\mathbf{B}^{\top}\mathbf{B})}}\right) \cdot s(\mathbf{B}) \log^2(n\kappa) \log(\epsilon^{-1})\right)$$

6 Reduction from High Probability Solvers to Expected Running Times

In this section we provide an auxiliary lemma that reduce the problem of achieving ϵ accuracy with high probability to the problem of achieving an accuracy c with probability at least δ for some constants c, δ up to logarithmic factor blowups. Note that a naive reduction suffers an additional $\log \log(1/\epsilon)$ blowup which we avoid in the following reduction. The reduction although straightforward helps us provide a concise description of the algorithm for the ERM problem in the next section.

Lemma 16. Consider being given a function $F: \mathbb{R}^d \to \mathbb{R}$ and define $x^* \stackrel{\text{def}}{=} \operatorname{argmin}_x F(x)$. Let \mathcal{A} be an algorithm such that given any point x_0 the algorithm runs in time \mathcal{T} and produces a point x' such that

$$F(x') - F(x^*) \le c (F(x_0) - F(x^*))$$

with probability at least $1 - \delta$ for given universal constants $c, \delta \in [0, 1]$. Further there exists a procedure \mathcal{P} which given a point x can produce an estimate m in time \mathcal{T}' such that

$$m/r < F(x) - F(x^*) < rm$$

for some given $r \geq 1$. Then there exists a procedure that given a point x_0 outputs a point x' such that

$$F(x') - F(x^*) \le \epsilon (F(x_0) - F(x^*))$$

and the expected running time of the procedure is bounded by

$$O\left((\mathcal{T} + \mathcal{T}')\log(r)\log(\epsilon^{-1})\right)$$

where O hides constant factors in c, δ . Moreover for any γ we have a procedure that produces a point x' such that

$$F(x') - F(x^*) \le \epsilon (F(x_0) - F(x^*))$$

with probability at least $1 - \gamma$ with a total running time of

$$O\left((\mathcal{T} + \mathcal{T}')\log(r)\log(\epsilon^{-1})\log(\gamma^{-1})\right)$$

The proof of the lemma is straightforward and we defer it to the Appendix (Section B).

7 Extension for ERM Problems

In this section we consider the ERM problem defined in Definition 6 and prove our main result (Theorem 7) for such problems. We propose Algorithm 4 coupled with Lemma 16 to solve the ERM case. The algorithm takes as input estimates of leverage scores of the matrix $\mathbf{A}^{\top}\mathbf{A}$. The algorithm then creates an estimator of the true function by sampling component functions according to the probability distribution given by the leverage scores and appropriate re-scaling to ensure unbiasedness. Further it reformulates the estimator as a sum of variance reduced components akin to [JZ13b]. The algorithm then approximately minimizes the estimator using an off-the-shelf ERM minimizer \mathcal{A} (in particular we use accelerated coordinate descent scheme of [AQRY16]). This step can be seen as analogous to the preconditioned iteration in the case of linear regression.

```
Algorithm 4: ERMSolve(x_0, \{\tau_i\}_{i=1}^n, F(x) = \sum_{i=1}^n f_i(x)), m)

Define for k = 1 \to n, p_k \stackrel{\text{def}}{=} \frac{\tau_k}{\sum_j \tau_j}.

Let \mathcal{D}(j) be the distribution over [1, \dots n] such that \forall k \ \Pr_{j \sim \mathcal{D}}(j = k) = p_k

Define for k = 1 \to n, \tilde{f}_k(x) \stackrel{\text{def}}{=} \frac{1}{p_k} \left[ f_k(x) - \nabla f_k(x_0)^\top x \right] + \nabla F(x_0)^\top x

Sample m integers i_1 \dots i_m \in [n] independently from \mathcal{D}.

if \sum_{t=1}^m \frac{\|a_{i_t}\|_2}{\sqrt{p_{i_t}}} \le 10m \sum_{k=1}^n \|a_k\|_2 \sqrt{p_k} then

\left| \begin{array}{c} \operatorname{Set} F_m(x) = \frac{1}{m} \sum_{t=1}^m \tilde{f}_{i_t}(x). \\ \operatorname{Use} \ \operatorname{Theorem} \ 18 \ \operatorname{to} \ \operatorname{find} \ x' \ \operatorname{such} \ \operatorname{that} \\ \end{array} \right|

end

Output: x'
```

Theorem 17 (given below) provides the decrease guarantee and bounds the running time of Algorithm 4.

Theorem 17. Given an ERM problem (Definition 6) and numbers u_i which are over estimates of leverage scores i.e. $u_i \ge \sigma_i$, set parameters such that $\tau_i = \min\{1, 20u_i \log(d)\}$, $m = 80\left((\sum_j \tau_j) \cdot M^4\right)\}$ then we have that Algorithm 4 produces a point x' such that

$$F(x') - \min F(x) \le \frac{1}{2} (F(x_0) - \min F(x))$$

with probability at least 1/2. Further Algorithm 4 can be implemented in total time

$$\tilde{O}\left(mM + \sum_{i=1}^{n} \frac{\|a_i\|_2 \sqrt{\tau_i} M^3}{\sqrt{\lambda_{\min}(\mathbf{A}^{\top} \mathbf{A})}}\right).$$

We now provide a quick proof sketch of Theorem 7 using Theorem 17.

Proof of Theorem 7. We make use of Lemma 16 plugging in Algorithm 4 as the procedure A. Note that c, δ are both 1/2 as guaranteed by Theorem 17. Moreover since F(x) is such that

that $\forall x \ M\lambda_{min}(\mathbf{A}^{\top}\mathbf{A}) \leq \nabla^2 F(x) \leq M\lambda_{max}(\mathbf{A}^{\top}\mathbf{A})$, we can use $\|\nabla F(x)\|_2^2$ as an estimator for $F(x) - F(x^*)$. The corresponding r for it is bounded by $M^2 \kappa(\mathbf{A}^{\top}\mathbf{A})$.

Note that the running time guaranteed in Theorem 17 depends on the quality of the estimates of the leverage scores input to it. We invoke Lemma 11 for computing accurate estimates of leverage scores. Putting together the above arguments finishes the proof for Theorem 7. \Box

The rest of the section is devoted to proving Theorem 17. We first provide a generalization of Theorem 4.

Theorem 18 (Acc. Coordinate Descent for general ERM). Consider the ERM problem as defined in Definition 6 with ϕ_i such that $\forall x \ \phi_i''(x) \in [\mu_i, L_i]$ and λ such that $\forall x \ \nabla^2 F(x) \succeq \lambda I$. Given a point x_0 there exists an algorithm \mathcal{A} which produces a point x' w.h.p in n such that

$$F(x') - \min F(x^*) \le \epsilon(F(x_0) - \min F(x^*))$$

in total time proportional to

$$\tilde{O}\left(\left(\sum_{i=1}^{n} \sqrt{\frac{L_i}{\mu_i}} + \sum_{i=1}^{n} \|a_i\|_2 \sqrt{\frac{L_i}{\lambda}}\right) s(\mathbf{A}) \log(\epsilon^{-1})\right)$$

The proof of Theorem 18 is a direct consequence of [AQRY16] and is deferred to the Appendix (Section C.1). We will use Algorithm \mathcal{A} as a subroutine in the Algorithm 4.

7.1 Proof of Theorem 17

Proof of Theorem 17. For convenience we restate the definitions provided in Algorithm 4. Given parameters $\{\tau_1 \dots \tau_n\}$ we define a probability distribution \mathcal{D} over $\{1, \dots n\}$ such that

$$\forall k \in [n] \ p_k \stackrel{\text{def}}{=} Pr_{j \sim \mathcal{D}}(j = k) \stackrel{\text{def}}{=} \frac{\tau_k}{\sum \tau_k} \quad . \tag{7.1}$$

We define approximations to f_k for $k \in [n]$ as

$$\tilde{f}_k(x) \stackrel{\text{def}}{=} \frac{1}{p_k} \left[f_k(x) - \nabla f_k(x_i)^T x \right] + \nabla F(x_i)^T x \quad . \tag{7.2}$$

As described in the algorithm we sample m integers $\{i_1, \ldots i_m\}$ independently from \mathcal{D} and we define an approximation to F

$$F_m(x) \stackrel{\text{def}}{=} \frac{1}{m} \sum_{t=1}^m \tilde{f}_{i_t}(x) \quad . \tag{7.3}$$

Further define

$$x_* = \operatorname{argmin}_x F(x)$$

In order to prove the theorem we will prove two key properties. Firstly the choice of the sample size $m = \Omega(\sum_{k=1}^{n} \tau_k M^4)$ is sufficient to ensure that approximately minimizing $F_m(x)$ is enough to make constant multiplicative factor progress. Further we will bound the running time of the inner coordinate descent procedure.

Consider the random matrix

$$\tilde{\mathbf{A}}^{\top} \tilde{\mathbf{A}} \stackrel{\text{def}}{=} \frac{1}{m} \sum_{t=1}^{m} \frac{a_{i_t} a_{i_t}^{\top}}{p_{i_t}} \quad . \tag{7.4}$$

Define the event \mathcal{E}_1 to be the following event.

$$\mathcal{E}_1 \stackrel{\text{def}}{=} \{ 0.5 \mathbf{A}^\top \mathbf{A} \leq \tilde{\mathbf{A}}^\top \tilde{\mathbf{A}} \leq 2 \mathbf{A}^\top \mathbf{A} \} \quad . \tag{7.5}$$

Via a direct application of the concentration inequality Lemma 24 (proved in the appendix Section C.2) we have that $Pr(\mathcal{E}_1) \geq 1 - 1/d$. The following lemma shows that under the above event we get a constant factor decrease in the error if we minimized F_m exactly.

Lemma 19. Consider an ERM problem $F(x) = \sum f_i(x)$ as defined in Definition 6. Let F_m be a sample of the ERM as defined in (7.3). Let $\tilde{\mathbf{A}}$ be as defined in (7.4). Let

$$x_m \stackrel{\text{def}}{=} \operatorname{argmin}_{x \in \mathbb{R}^d} F_m(x).$$

Let $\mathcal{E}_1 \stackrel{\text{def}}{=} \{0.5\mathbf{A}^{\top}\mathbf{A} \leq \tilde{\mathbf{A}}^{\top}\tilde{\mathbf{A}} \leq 2\mathbf{A}^{\top}\mathbf{A}\}$ and let $Pr(\mathcal{E}_1) \geq p$. Then if we set $m \geq 80(\sum_j \tau_j) \cdot M^4$, we have that

$$\Pr\left(F(x_m) - F(x_*) \le O\left(\frac{1}{4}\left(F(x_0) - F(x_*)\right)\right)\right) \ge p - \frac{1}{10}$$
(7.6)

The above lemma bounds the number of samples required for sufficient decrease per outer iteration. For the rest of the proof we will assume that the event \mathcal{E}_1 and the property (7.6) holds. Note by Lemma 19 that this happens with probability at least 7/10. Further note that the probability that the condition in the if loop, i.e.

$$\sum_{t=1}^{m} \frac{\|a_{i_t}\|_2}{\sqrt{p_{i_t}}} \le 10m \sum_{k=1}^{n} \|a_k\|_2 \sqrt{p_k}$$
(7.7)

happens is at least 9/10. This is a direct implication of Markov's inequality and the fact that

$$\mathbb{E}\left[\sum_{t=1}^{m} \frac{\|a_{i_t}\|_2}{\sqrt{p_{i_t}}}\right] = m \sum_{k=1}^{n} \|a_k\|_2 \sqrt{p_k}$$

Putting the above together via a simple union bound gives us that with probability at least 6/10 all three of the following happen \mathcal{E}_1 , Condition (7.6) and the execution of the if loop (i.e. Condition 7.7 is met).

We now show that under the above conditions we get sufficient decrease in x'. Firstly note that by definition we have that

$$F_m(x') - F_m(x_m) \le \frac{1}{512M^4} \left(F_m(x_0) - F_m(x_m) \right)$$
 (7.8)

Note that if event \mathcal{E}_1 happens then

$$\forall x \quad \frac{1}{2M} \mathbf{A}^{\top} \mathbf{A} \le \nabla^2 F_m(x) \le 2M \mathbf{A}^{\top} \mathbf{A} \quad . \tag{7.9}$$

Now consider the RHS of (7.8)

$$F_{m}(x_{0}) - F_{m}(x_{m}) \leq M \|x_{0} - x_{m}\|_{\mathbf{A}^{\top}\mathbf{A}}^{2}$$

$$\leq 2M(\|x_{0} - x_{*}\|_{\mathbf{A}^{\top}\mathbf{A}}^{2} + \|x_{m} - x_{*}\|_{\mathbf{A}^{\top}\mathbf{A}}^{2})$$

$$\leq 4M^{2}(F(x_{0}) - F(x_{*}) + F(x_{m}) - F(x_{*}))$$

$$\leq 5M^{2}(F(x_{0}) - F(x_{*}))$$

$$(7.10)$$

The first inequality follows from Equation 7.9, second from triangle inequality, third by noting that F is $\frac{1}{M}$ strongly convex in the norm given by $\mathbf{A}^{\top}\mathbf{A}$ (Assumption (1.4)) and the fourth from Lemma 19. Further note that

$$F(x') - F(x_m) \leq \nabla F(x_m)^{\top} (x' - x_m) + \frac{M}{2} \|x' - x_m\|_{\mathbf{A}^{\top} \mathbf{A}}^{2}$$

$$\leq \|\nabla F(x_m)^{\top}\|_{[\mathbf{A}^{\top} \mathbf{A}]^{-1}} \|(x' - x_m)\|_{\mathbf{A}^{\top} \mathbf{A}} + \frac{M}{2} \|x' - x_m\|_{\mathbf{A}^{\top} \mathbf{A}}^{2}$$

$$\leq \sqrt{2M(F(x_m) - F(x_*))} \|(x' - x_m)\|_{\mathbf{A}^{\top} \mathbf{A}} + \frac{M}{2} \|x' - x_m\|_{\mathbf{A}^{\top} \mathbf{A}}^{2}$$

$$\leq \sqrt{2M(F(x_m) - F(x_*))} \sqrt{4M(F_m(x') - F_m(x_m))} + 2M^2(F_m(x') - F_m(x_m))$$

$$\leq \frac{1}{8M} \sqrt{(F(x_m) - F(x_*))} \sqrt{(F_m(x_0) - F_m(x_m))} + \frac{1}{256M^2} (F_m(x_0) - F_m(x_m))$$

$$\leq \frac{1}{3} (F(x_0) - F(x_*))$$

$$(7.11)$$

The first and third inequality follow by noting that F is M smooth and 1/M strongly convex in $\mathbf{A}^{\top}\mathbf{A}$ norm. Fourth inequality follows by noting that if event \mathcal{E}_1 holds F_m is 1/2M strongly convex in $\mathbf{A}^{\top}\mathbf{A}$ norm. Fifth inequality follows from (7.8) and sixth inequality follows from (7.10).

(7.11) together with (7.6) implies that with probability at least 6/10, we have that

$$F(x') - F(x_*) \le \frac{1}{2}(F(x_0) - F(x_*))$$

We will now bound the running time of the procedure via Theorem 18. Define L_{i_t} and μ_{i_t} to respectively the smoothness and strong convexity parameters of the components $\frac{\tilde{f}_{i_t}}{m}$. Note that $L_{i_t} \leq \frac{M}{mp_{i_t}}$ and $\mu_{i_t} \geq \frac{1}{Mmp_{i_t}}$. Note that event \mathcal{E}_1 gives us that $\forall x \nabla^2 F_m(x) \succeq \frac{1}{2M} \lambda_{min}(\mathbf{A}^{\top} \mathbf{A})$. A direct application of Theorem 18 using the bounds on L_{i_t} and μ_{i_t} gives us that the total running time is bounded by

$$\tilde{O}\left(\left(\sum_{t=1}^{m} M + \sum_{t=1}^{m} \|a_{i_t}\|_2 \sqrt{\frac{M^2}{mp_{i_t}}}\right) s(\mathbf{A}) \log(\epsilon^{-1})\right) \leq \tilde{O}\left(mM + \sum_{i=1}^{n} \frac{\|a_i\|_2 \sqrt{\tau_i} M^3}{\sqrt{\lambda_{\min}(\mathbf{A}^{\top} \mathbf{A})}}\right)$$

The inequality follows from Condition (7.7) and the definitions of p_k, m .

7.2 Proof of Lemma 19

Proof of Lemma 19. Consider the definitions in (7.1), (7.2), (7.3). Note the following easy observation.

$$F(x) = \mathbb{E}_{k \sim \mathcal{D}} \tilde{f}_k(x)$$

Consider the following Lemma 20 which connects the optima of two convex functions F and G.

Lemma 20. Let $F(x), G(y) : \mathbb{R}^n \to R$ be twice differentiable and strictly convex. Define

$$x_* = \operatorname{argmin}_x F(x)$$
 and $y_* = \operatorname{argmin}_y G(y)$

Then we have that

$$F(y_*) - F(x_*) = \|\nabla G(x_*)\|_{\mathbf{H}_C^{-1}\mathbf{H}_F\mathbf{H}_C^{-1}}^2$$
.

where $\mathbf{H}_F \stackrel{\text{def}}{=} \int_0^1 \nabla^2 F(t.y_* + (1-t)x_*) dt$ and $\mathbf{H}_G \stackrel{\text{def}}{=} \int_0^1 \nabla^2 G(t.y_* + (1-t)x_*)$.

We wish to invoke Lemma 20 by setting F = F(x), $G = F_m(x)$. In this setting we have that

$$\mathbf{H}_F \stackrel{\text{def}}{=} \int_0^1 \nabla^2 F(t.x_m + (1-t)x_*) dt \text{ and } \mathbf{H}_G \stackrel{\text{def}}{=} \int_0^1 \nabla^2 F_m(t.x_m + (1-t)x_*)$$

Firstly note that the definition of F and Assumption (1.4) gives us that

$$\mathbf{H}_F \leq M \cdot \mathbf{A}^{\top} \mathbf{A} \tag{7.12}$$

Using Definition 7.4 and Assumption (1.4) gives us that

$$\mathbf{H}_G \preceq M \cdot \tilde{\mathbf{A}}^{\top} \tilde{\mathbf{A}}$$

Combining the above two and noting that the event \mathcal{E}_1 happens with probability at least p we get that

$$\mathbf{H}_G^{-1} \mathbf{A}^{\mathsf{T}} \mathbf{A} \mathbf{H}_G^{-1} \leq 2M^2 [\mathbf{A}^{\mathsf{T}} \mathbf{A}]^{-1} \text{ w.p. } p$$

$$(7.13)$$

Also note that for any fixed matrix \mathbf{R} , we have that

$$\mathbb{E}[\|\nabla F_m(x_*)\|_{\mathbf{R}}^2] = \frac{\mathbb{E}_{k \sim \mathcal{D}}[\|\nabla \tilde{f}_k(x_*)\|_{\mathbf{R}}^2]}{m}$$

which implies via Markov's inequality that with probability at least 9/10 we have that

$$\|\nabla F_m(x_*)\|_{\mathbf{R}}^2 \le \frac{10\mathbb{E}_{k \sim \mathcal{D}}[\|\nabla \tilde{f}_k(x_*)\|_{\mathbf{R}}^2]}{m}$$

$$(7.14)$$

Putting (7.13) and (7.14) together and using a union bound we get that

$$||F_m(x_*)||_{\mathbf{H}_G^{-1}\mathbf{A}^{\top}\mathbf{A}\mathbf{H}_G^{-1}}^2 \le \frac{20M^2 \mathbb{E}_{k \sim \mathcal{D}}[||\nabla \tilde{f}_k(x_*)||_{[\mathbf{A}^{\top}\mathbf{A}]^{-1}}^2]}{m} \text{ w.p. } p - 1/10$$

Using Lemma 20 and (7.12) we get that with probability at least p-1/10

$$F(x_m) - F(x_*) \le \frac{20M^3 \mathbb{E}_{k \sim \mathcal{D}}[\|\nabla \hat{f}_k(x_*)\|_{[\mathbf{A}^\top \mathbf{A}]^{-1}}^2]}{m}$$
(7.15)

We will now connect $\mathbb{E}_{k \sim \mathcal{D}}[\|\nabla \tilde{f}_k(x_*)\|_{[\mathbf{A}^\top \mathbf{A}]^{-1}}^2]$ with the error at x_i .

Lemma 21. Consider an ERM function $F(x) = \sum_{i=1}^m f_i(x)$ where $f_i(x) = \psi_i(a_i^\top x)$ with $\psi_i'' \in [\frac{1}{M}, M]$. Define a distribution $\mathcal{D}(j)$ over [n] such that $Pr(j = k) = p_k \stackrel{\text{def}}{=} \frac{\tau_k}{\sum \tau_k}$ for numbers $\tau_k \stackrel{\text{def}}{=} \min(1, 20u_k \log(d))$ where $u_k \geq \sigma_i(\mathbf{A})^6$ are overestimates of leverage scores. Given a point \tilde{x} consider the variance reduced reformulation

$$F(x) = \mathbb{E}_{k \sim D}[\tilde{f}_k(x)]$$

where

$$\tilde{f}_k(x) \stackrel{\text{def}}{=} \frac{1}{p_k} \left[f_k(x) - \nabla f_k(\tilde{x})^\top x \right] + \nabla F(\tilde{x})^\top x$$

Then we have that

$$\mathbb{E}_{k \sim D} \left[\|\nabla \tilde{f}_k(x_*)\|_{[\mathbf{A}^\top \mathbf{A}]^{-1}}^2 \right] \le 2(\sum_j \tau_j) \cdot M \cdot (F(\tilde{x}) - F(x_*))$$

Putting together (7.15) and Lemma 21 we get that

$$F(x_m) - F(x_*) \le O\left(\frac{(\sum_j \tau_j) \cdot M^4}{m} \cdot (F(x_0) - F(x_*))\right) \text{ w.p } p - \frac{1}{10}$$

Lemma 19 now follows from the choice of m.

We finish this section with proofs of Lemma 20 and 21

Proof of Lemma 20. For all $t \in [0,1]$ let $z(t) \stackrel{\text{def}}{=} t \cdot y_* + (1-t) \cdot x_*$ for $t \in [0,1]$ and $\mathbf{H}_F \stackrel{\text{def}}{=} \int_0^1 \nabla^2 F(z(t)) dt$. By Taylor series expansion we have that

$$F(x_n) = F(x_*) + \nabla F(x_*)^{\top} (x_n - x_*) + \int_0^1 \frac{1}{2} (x_n - x_*)^{\top} \nabla^2 F(z(t)) (x_n - x_*) dt$$

= $F(x_*) + \frac{1}{2} ||x_n - x_*||_{\mathbf{H}_F}^2$.

Here we used that $\nabla F(x_*) = 0$ and $\nabla^2 F(z(t)) \succeq \mathbf{0}$ by the convexity of F. We also have by definition that

$$\nabla G(y_*) = \vec{0}$$

and therefore

$$\nabla G(y_*) - \nabla G(x_*) = \int_0^1 \nabla^2 G(z(t))(y_* - x_*) \cdot dt$$

and

$$(y_* - x_*) = -\mathbf{H}_G^{-1} \nabla G(x_*)$$

where $\mathbf{H}_G \stackrel{\text{def}}{=} \int_0^1 \nabla^2 G(z(t))$. We now have that

$$F(y_*) - F(x_*) = \frac{1}{2} \|y_* - x_*\|_{\mathbf{H}_F}^2 = \|\nabla G(x_*)\|_{\mathbf{H}_G^{-1}\mathbf{H}_F\mathbf{H}_G^{-1}}^2$$
 (7.16)

 $^{^6\}sigma_i$ are leverage scores defined in Definition 8

Proof of Lemma 21. For the purpose of this proof it will be convenient to perform a change of basis. Define the function

$$G(x) = \mathbb{E}_{k \sim \mathcal{D}} g_i(x) \text{ where } g_i(x) = \frac{1}{p_i} f_k((\mathbf{A}^\top \mathbf{A})^{-1/2} x)$$

Note that $G(x) = F((\mathbf{A}^{\top}\mathbf{A})^{-1/2})x$). We will first note that

$$\nabla^2 g_i(x) = \frac{1}{p_i} \cdot \left[(\mathbf{A}^\top \mathbf{A})^{-1/2} a_i a_i^\top (\mathbf{A}^\top \mathbf{A})^{-1/2} \cdot \psi_i'' (a_i^\top (\mathbf{A}^\top \mathbf{A})^{-1/2} x) \right]$$

and now by the cyclic property of trace and the fact that $\psi_i'' \leq M$ we have

$$\operatorname{tr}(\nabla^2 g_i(x)) = \frac{(\sum_j \tau_j)}{\tau_i} \cdot a_i^{\top} (\mathbf{A}^{\top} \mathbf{A})^{-1} a_i \cdot M$$

Note that $a_i^{\top}(\mathbf{A}^{\top}\mathbf{A})^{-1}a_i \leq 1$. Now either $\tau_i = 1$ or $\tau_i = 20a_i^{\top}(\mathbf{A}^{\top}\mathbf{A})^{-1}a_i\log(d)$. In both cases we see that RHS above ≤ 1 . Therefore we get that g_i is $(\sum_j \tau_j)M$ smooth. We now have the following lemma.

Lemma 22. Let \mathcal{D} be any distribution over [n] and define $g = \mathbb{E}_{i \sim \mathcal{D}}[g_i(x)]$ for component convex functions g_i each of which is L smooth. Let $x_* \stackrel{\text{def}}{=} \operatorname{argmin} g(x)$. We have that

$$\mathbb{E}_{i \sim D} \|\nabla g_i(x) - \nabla g_i(x_*)\|_2^2 \le 2L(g(x) - g(x^*))$$

The proof of the above Lemma is identical to the proof of Equation 8 in [JZ13b] and is provided in the appendix (Section C.3) for completeness. Now note that

$$2(\sum_{j} \tau_{j}) M(f(\tilde{x}) - f(x_{*})) = 2(\sum_{j} \tau_{j}) \cdot M \cdot (g((\mathbf{A}^{\top} \mathbf{A})\tilde{x}) - g((\mathbf{A}^{\top} \mathbf{A})x_{*}))$$

$$\geq \mathbb{E}_{i \sim \mathcal{D}} \|\nabla g_{i}((\mathbf{A}^{\top} \mathbf{A})\tilde{x}) - \nabla g_{i}((\mathbf{A}^{\top} \mathbf{A})x_{*})\|_{2}^{2}$$

$$= \mathbb{E}_{i \sim \mathcal{D}} \|(\mathbf{A}^{\top} \mathbf{A})^{-1/2} \frac{1}{p_{i}} (\nabla f_{i}(\tilde{x}) - \nabla f_{i}(x_{*}))\|_{2}^{2}$$

$$= \mathbb{E}_{i \sim \mathcal{D}} \|\nabla \tilde{f}_{i}(x_{*})\|_{(\mathbf{A}^{\top} \mathbf{A})^{-1}}^{2} - 2\mathbb{E}_{i \sim \mathcal{D}} \left[\frac{1}{p_{i}} (\nabla f_{i}(x_{*}) - \nabla f_{i}(\tilde{x}))^{\top} [\mathbf{A}^{\top} \mathbf{A}]^{-1} \nabla F(\tilde{x}) \right]$$

$$- \|\nabla F(\tilde{x})\|_{[\mathbf{A}^{\top} \mathbf{A}]^{-1}}^{2}$$

$$\geq \mathbb{E}_{i \sim \mathcal{D}} \|\nabla \tilde{f}_{i}(x_{*})\|_{(\mathbf{A}^{\top} \mathbf{A})^{-1}}^{2}$$

The first line follows by definition. The second line by 22 and by noting that g is $(\sum_j \tau_j)M$ smooth. The third and fourth line follows by definition. The fifth line follows by noting that $\nabla F(x_*) = 0$. \square

References

[All16] Zeyuan Allen Zhu. Katyusha: Accelerated variance reduction for faster SGD. CoRR, abs/1603.05953, 2016.

- [AQRY16] Zeyuan Allen Zhu, Zheng Qu, Peter Richtárik, and Yang Yuan. Even faster accelerated coordinate descent using non-uniform sampling. In *Proceedings of the 33nd International Conference on Machine Learning, ICML 2016, New York City, NY, USA, June 19-24, 2016*, pages 1110–1119, 2016.
- [CLM+15] Michael B. Cohen, Yin Tat Lee, Cameron Musco, Christopher Musco, Richard Peng, and Aaron Sidford. Uniform sampling for matrix approximation. In Proceedings of the 2015 Conference on Innovations in Theoretical Computer Science, ITCS 2015, Rehovot, Israel, January 11-13, 2015, pages 181-190, 2015.
- [Coh16] Michael B. Cohen. Nearly tight oblivious subspace embeddings by trace inequalities. In *Proceedings of the Twenty-Seventh Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2016, Arlington, VA, USA, January 10-12, 2016*, pages 278–287, 2016.
- [CW13] Kenneth L. Clarkson and David P. Woodruff. Low rank approximation and regression in input sparsity time. In *Symposium on Theory of Computing Conference*, STOC'13, Palo Alto, CA, USA, June 1-4, 2013, pages 81–90, 2013.
- [FGKS15a] Roy Frostig, Rong Ge, Sham Kakade, and Aaron Sidford. Un-regularizing: approximate proximal point and faster stochastic algorithms for empirical risk minimization. In Proceedings of the 32nd International Conference on Machine Learning, ICML 2015, Lille, France, 6-11 July 2015, pages 2540–2548, 2015.
- [FGKS15b] Roy Frostig, Rong Ge, Sham M Kakade, and Aaron Sidford. Competing with the empirical risk minimizer in a single pass. In *COLT*, pages 728–763, 2015.
- [Har12] Nick Harvey. Matrix concentration, 2012.
- [JZ13a] Rie Johnson and Tong Zhang. Accelerating stochastic gradient descent using predictive variance reduction. In Advances in Neural Information Processing Systems 26: 27th Annual Conference on Neural Information Processing Systems 2013. Proceedings of a meeting held December 5-8, 2013, Lake Tahoe, Nevada, United States., pages 315–323, 2013.
- [JZ13b] Rie Johnson and Tong Zhang. Accelerating stochastic gradient descent using predictive variance reduction. In *Advances in Neural Information Processing Systems*, pages 315–323, 2013.
- [KSST09] S Kakade, Shai Shalev-Shwartz, and Ambuj Tewari. Applications of strong convexity—strong smoothness duality to learning with matrices. 2009.
- [LMH15] Hongzhou Lin, Julien Mairal, and Zaïd Harchaoui. A universal catalyst for first-order optimization. In Advances in Neural Information Processing Systems 28: Annual Conference on Neural Information Processing Systems 2015, December 7-12, 2015, Montreal, Quebec, Canada, pages 3384–3392, 2015.
- [LMP13] Mu Li, Gary L. Miller, and Richard Peng. Iterative row sampling. In 54th Annual IEEE Symposium on Foundations of Computer Science, FOCS 2013, 26-29 October, 2013, Berkeley, CA, USA, pages 127–136, 2013.

- [Nes83] Yu. Nesterov. A method for solving a convex programming problem with convergence rate 1/k^2. Doklady AN SSSR, 269:543–547, 1983.
- [NN13] Jelani Nelson and Huy L. Nguyen. OSNAP: faster numerical linear algebra algorithms via sparser subspace embeddings. In 54th Annual IEEE Symposium on Foundations of Computer Science, FOCS 2013, 26-29 October, 2013, Berkeley, CA, USA, pages 117–126, 2013.
- [NS17] Yurii Nesterov and Sebastian U. Stich. Efficiency of the accelerated coordinate descent method on structured optimization problems. SIAM Journal on Optimization, 27(1):110–123, 2017.
- [SS08] Daniel A. Spielman and Nikhil Srivastava. Graph sparsification by effective resistances. In Proceedings of the 40th Annual ACM Symposium on Theory of Computing, Victoria, British Columbia, Canada, May 17-20, 2008, pages 563–568, 2008.
- [SSZ13] Shai Shalev-Shwartz and Tong Zhang. Stochastic dual coordinate ascent methods for regularized loss. *J. Mach. Learn. Res.*, 14(1):567–599, February 2013.
- [SZ16] Shai Shalev-Shwartz and Tong Zhang. Accelerated proximal stochastic dual coordinate ascent for regularized loss minimization. *Math. Program.*, 155(1-2):105–145, 2016.
- [Wil12] Virginia Vassilevska Williams. Multiplying matrices faster than coppersmith-winograd. In Proceedings of the 44th Symposium on Theory of Computing Conference, STOC 2012, New York, NY, USA, May 19 22, 2012, pages 887–898, 2012.
- [WS16] Blake E Woodworth and Nati Srebro. Tight complexity bounds for optimizing composite objectives. In *Advances in Neural Information Processing Systems*, pages 3639–3647, 2016.

A Computing Leverage Scores given a Regression Algorithm

In this section we give a proof of Lemma 11 which bounds the running time of computing leverage scores assuming access to a regression algorithm. The main algorithm is given as Algorithm 2.

Proof of Lemma 11. Let $y_j^* = (\mathbf{A}^\top \mathbf{A})^{-1} \mathbf{A}^\top v_j$ be the minimizer of $f_{\mathbf{A},v_j}(x)$. (1.2) shows that

$$\|\mathbf{A}y_j - \mathbf{A}y_j^*\|_2^2 \le \epsilon \cdot v_j^{\mathsf{T}} \mathbf{A} (\mathbf{A}^{\mathsf{T}} \mathbf{A})^{-1} \mathbf{A}^{\mathsf{T}} v_j.$$

Using $v_i \sim N(0, I)$, we have that

$$v_j^{\top} \mathbf{A} (\mathbf{A}^{\top} \mathbf{A})^{-1} \mathbf{A}^{\top} v_j \le 2d \cdot \log(n)$$

with probability $1 - n^{-\Theta(1)}$. Hence, we have that

$$\left| e_i^{\top} \mathbf{A} y_j - e_i^{\top} \mathbf{A} y_j^* \right| \le \|\mathbf{A} y_j - \mathbf{A} y_j^*\|_2 \le \sqrt{2\epsilon d \cdot \log(n)}.$$

Using this and

$$\left| e_i^\top \mathbf{A} y_j^* \right| \leq \sqrt{e_i^\top \mathbf{A} (\mathbf{A}^\top \mathbf{A})^{-1} \mathbf{A}^\top e_i} \sqrt{v_j^\top \mathbf{A} (\mathbf{A}^\top \mathbf{A})^{-1} \mathbf{A}^\top v_j} \leq \sqrt{2d \cdot \log(n)} ,$$

we have that

$$\left| \left(e_i^{\top} \mathbf{A} y_j \right)^2 - \left(e_i^{\top} \mathbf{A} y_j^* \right)^2 \right| \le 6\sqrt{\epsilon} d \cdot \log(n).$$

Using the definition of ϵ , we have that

$$\left| \frac{1}{k} \sum_{j=1}^{k} (e_i^{\top} \mathbf{A} y_j)^2 - \frac{1}{k} \sum_{j=1}^{k} (e_i^{\top} \mathbf{A} y_j^*)^2 \right| \le 6\sqrt{\epsilon} d \cdot \log(n) \le \frac{\delta}{3n \cdot \kappa(\mathbf{A}^{\top} \mathbf{A})}$$
(A.1)

Also, we note that

$$\frac{1}{k} \sum_{i=1}^k (e_i^{\top} \mathbf{A} y_j^*)^2 = \frac{1}{k} \sum_{i=1}^k (e_i^{\top} \mathbf{A} (\mathbf{A}^{\top} \mathbf{A})^{-1} \mathbf{A}^{\top} v_j)^2.$$

Since $v_j \sim N(0, I)$ and $k = c \log(n)/\delta^2$ where c is some large enough constant, Johnson-Lindenstrauss lemma shows that, with high probability in n for all $i \in [n]$

$$\left(1 - \frac{\delta}{3}\right)\sigma_i(\mathbf{A}) \le \frac{1}{k} \sum_{i=1}^k (e_i^\top \mathbf{A} y_j^*)^2 \le \left(1 + \frac{\delta}{3}\right)\sigma_i(\mathbf{A})$$

Combining this with (A.1) gives the result.

Finally, to check the success probability of this algorithm, we note that we solved $O(\delta^{-2} \log n)$ many regression problems and each one has success probability $1 - n^{-\Theta(1)}$. Also, the Johnson–Lindenstrauss lemma succeed with probability $1 - n^{-\Theta(1)}$. This gives the result.

B Reductions between High Probability and Expected Running Time

B.1 Proof of Lemma 16

Proof of Lemma 16. To show the lemma we will show the existence of a procedure (described in Algorithm 5) which produces a point x' such that

$$F(x') - F(x^*) \le 1/2 \left(F(x_0) - F(x^*) \right) \tag{B.1}$$

with expected running time bounded by $O((\mathcal{T} + \mathcal{T}')\log(r))$. Applying this procedure $O(\log(\epsilon^{-1}))$ and using linearity of expectation gives us the Lemma 16. Consider the following procedure to prove Lemma 16.

Note that since for every x_{ij} we have that

$$F(x_{ij}) - F(x^*) \le c (F(x_i) - F(x^*))$$

with probability at least δ , therefore we have that

$$F(x_{i+1}) - F(x^*) \le c (F(x_i) - F(x^*))$$

Algorithm 5: Reduction $(x_0, F(x), \mathcal{P}, \mathcal{A}, c, \delta, r)$

with probability at least $1 - \delta^{\log_{\delta^{-1}}(2\log_{c^{-1}}(r^2))} = 1 - \frac{1}{2\log_{c^{-1}}(r^2)}$. Taking a union bound over the outer loop gives us that with probability at least 1/2 we have that

$$F(x_T) - F(x^*) \le \frac{1}{2r^2} \left(F(x_i) - F(x^*) \right)$$

Moreover by the property of the estimates given by \mathcal{P} we know that in this case we have that $E \leq 0.5$. Therefore we have that with probability at least 1/2 the repeat loop computes an x_T that reduces error by at least a factor of 1/2 and we can verify it. Therefore in expectation the loop runs a total of 2 times. The total runtime of the above procedure can easily seen to be $O\left((\mathcal{T} + \mathcal{T}')\log(r)\log(\epsilon^{-1})\right)$.

Further suppose we are given a procedure with the guarantee that for any ϵ in expected running time \mathcal{T}_{ϵ} it produces a point x' such that

$$F(x') - \min F(x) < \epsilon(F(x_0) - \min F(x))$$

We now run this procedure for time $\mathcal{T}_{\epsilon/2}$. By Markov's inequality with probability at least 1/2 we have a point that satisfies

$$F(x') - \min F(x) \le \epsilon(F(x_0) - \min F(x))$$

It is now easy to see that if we repeat the above procedure $\log(\gamma^{-1})$ many times and take the x with the minimum value we have a point x' such that

$$F(x') - \min F(x) \le \epsilon(F(x_0) - \min F(x))$$

with probability at least $1 - \gamma$.

C Proofs and Theorems from the Generalized ERM Section

C.1 Accelerated Coordinate Descent for ERM

Proof of Theorem 18. To remind the reader

$$f(x) = \sum_{i=1}^{n} \psi_i(a_i^{\top} x)$$
 where $\psi_i''(x) \in [\mu_i, L_i]$.

Following is a well known theorem. For a proof see [KSST09].

Theorem 23 (Strong / Smooth Duality). A closed and convex function f is β -strongly convex with respect to a norm $\|\cdot\|$ if and only if f^* is $\frac{1}{\beta}$ -strongly smooth w.r.t the dual norm of $\|\cdot\|$.

A direct application of the above theorem gives us that $\psi_i^{*''}(y) \in [\frac{1}{L_i}, \frac{1}{\mu_i}]$. Consider the function

$$g_s(y) = \sum_{i=1}^n \psi_i^*(y_i) + \frac{1}{2\lambda} \|\mathbf{A}^\top y\|_2^2 - s^\top \mathbf{A}^\top y$$

Consider the following modified function $\tilde{g}_s(y) \stackrel{\text{def}}{=} g_s(\mathbf{D}y)$ where \mathbf{D} is a diagonal matrix with $\mathbf{D}_{ii} = L_i$. We will equivalently minimize the function $\tilde{g}_s(y_i)$. We now immediately get that the function $\tilde{g}_s(y)$ is 1 strongly convex. Moreover we have that

$$\frac{d^2}{dy_i^2}g_s(y) = \frac{L_i}{\mu_i} + \frac{1}{\lambda} ||a_i||^2 L_i .$$

Hence, Theorem 14 finds y satisfying (5.1) in time

$$O\left(s(\mathbf{A}) \cdot \sum_{i \in [n]} \sqrt{\frac{L_i}{\mu_i} + \frac{1}{\lambda} \|a_i\|^2 L_i \log(\epsilon^{-1})}\right) = O\left(\left(\sum_{i=1}^n \sqrt{\frac{L_i}{\mu_i}} + \frac{1}{\sqrt{\lambda}} \sum_{i=1}^n \|a_i\| \sqrt{L_i}\right) s(\mathbf{A}) \log(\epsilon^{-1})\right)$$

A direct application of Theorem 15 gives that the total running time is

$$O\left(\left(\sum_{i=1}^{n} \sqrt{\frac{L_i}{\mu_i}} + \frac{1}{\sqrt{\lambda}} \sum_{i=1}^{n} \|a_i\| \sqrt{L_i}\right) s(\mathbf{A}) \log(n\kappa) \log(\kappa/\epsilon)\right)$$

The above equation assumes that the inner iterations of accelerated coordinate descent can be implemented in $O(s(\mathbf{A}))$. This is easy to see because diagonal scaling is linear in sparsity. Therefore the only bottleneck is computing the gradient of the dual function ψ^* . We can assume that ψ is explicit and therefore the gradient of ψ^* is easily computed.

C.2 A Matrix Concentration Inequality for Sampling with Replacement

Lemma 24. Given an error parameter $0 \le \epsilon \le 1$, let u be a vector of leverage score overestimates, i.e. $\sigma_i(\mathbf{A}) \le u_i$ for all i. Let $\alpha = \epsilon^{-2}$ be a sampling rate parameter and c be a fixed constant. For each row we define a number $\gamma_i = \min\{1, \alpha c u_i \log(d)\}$ and a probability $p_i = \frac{\gamma_i}{\sum \gamma_i}$. Let Y_j be a random variable which is sampled by picking a vector a_i with probability p_i and setting $Y_j = \frac{a_i a_i^\top}{p_i}$. Now consider the random variable $Y = \frac{1}{m} \sum_j Y_j$. We have that as long as $m \ge \sum_i \gamma_i$ then

$$\Pr((1-\epsilon)\mathbf{A}^{\top}\mathbf{A} \leq Y \leq (1+\epsilon)\mathbf{A}^{\top}\mathbf{A}) \geq 1 - d^{-c/3}$$

Proof. The proof of the lemma follows the proof of Lemma 4 in [CLM⁺15]. We only state the differences. We use the inequality given in [Har12].

Lemma 25. Let $Y_1 ... Y_k$ be independent random positive semidefinite matrices of size $d \times d$. Let $Y = \sum Y_i$ and let $Z = \mathbb{E}[Y]$. If $Y_i \leq R.Z$ then

$$\Pr\left[\sum Y_i \preceq (1-\epsilon)Z\right] \leq de^{-\frac{\epsilon^2}{2R}} \quad and \quad \Pr\left[\sum Y_i \succeq (1+\epsilon)Z\right] \leq de^{-\frac{\epsilon^2}{3R}} \ .$$

Note that the expectation of $Y_j/m = a_i a_i^{\top}/m$. Moreover note that each

$$\frac{Y_j}{m} \preceq \max_i \frac{a_i a_i^\top \sum_k \gamma_k}{m \gamma_i} \preceq \frac{\mathbf{A}^\top \mathbf{A}}{c \log d \epsilon^{-2}}$$

The inequality follows from noting that $m \ge \sum \gamma_i$ and Equation 10 in [CLM⁺15]. The calculations now follow exactly in the same way as in the proof in [CLM⁺15].

C.3 Proof of Lemma 22

Proof of Lemma 22. Let $x_* \stackrel{\text{def}}{=} \operatorname{argmin} g(x)$. Define auxiliary functions

$$h_i(x) \stackrel{\text{def}}{=} g_i(x) - g_i(x_*) - \nabla g_i(x_*)^{\top} (x - x_*)$$

We know that $h_i(x_*) = \min h_i(x)$ since $\nabla h_i(x_*) = 0$. Using smoothness of h and that $h_i(x_*) = 0$, we now have that

$$\|\nabla h_i(x)\|_2^2 \le 2Lh_i(x)$$

A simple substitution gives us that for all i

$$\|\nabla g_i(x) - \nabla g_i(x_*)\|_2^2 \le 2L \left(g_i(x) - g_i(x_*) - \nabla g_i(x_*)^\top (x - x_*)\right)$$

Taking expectations and using the fact that $g(x_*) = 0$ gives us that

$$\mathbb{E}_{i \sim D} \|\nabla g_i(x) - \nabla g_i(x_*)\|_2^2 \le 2L(g(x) - g(x^*))$$